GCSE Maths to A Level Maths

Transition Booklet

The most popular choice at A level is Mathematics; however, many students say that they find the initial transition from GCSE to A level challenging. There are a certain set of skills that are imperative when making the transition for GCSE to A Level Maths

These resources are designed for you to complete independently and will develop fluency in the fundamental techniques and the key mathematical concepts that underpin A level Mathematics.

The link below contains videos for you to watch, along with a downloadable PDF of example questions, **practice questions** and End of chapter exam questions for consolidation.

Free videos to assist the transition from GCSE to A level Maths | Pearson UK

	TOPIC
1	Expanding brackets and simplifying expressions
2	Surds
3	Rules of indices
4	Factorising expressions
5	Completing the square
6	Solving quadratic equations
7	Sketching quadratic graphs
8	Solving linear simultaneous equations
9	Solving linear and quadratic simultaneous equations
10	Solving simultaneous equations graphically
11	Linear inequalities
12	Quadratic inequalities
13	Sketching cubic and reciprocal graphs
14	Translating graphs.

You will need to be very organised so keep your work in a folder & mark any queries to ask at the beginning of term. In the **first week of term you will take a test** to check how well you understand these topics, so it is important that you have completed the booklet by then. The pass mark is 75%. If you do not pass this test, you may be asked to leave the course or to complete a programme of additional work in order to bring your basic algebra skills to the required standard

A Level Maths & Further Maths

The content of the Mathematics A-level is taught to allow progression on to some very interesting and challenging areas of Mathematics. Two thirds of the Mathematics course is Pure Mathematics with the other third being split evenly between Mechanics and Statistics.

50% of the Further Mathematics course is Pure Mathematics, allowing for learning the applied modules of Further Mechanics and Decision Mathematics.

Entry Requirement - **A Level Maths – At least a grade 7 in GCSE**
Further Maths – At least a grade 8 in GCSE

Assessment structure

A level Mathematics – Exam Board Edexcel	
Paper 1: Pure Mathematics 33%, 2 hours, 100 marks	Any A level pure mathematics content can be assessed on either paper
Paper 2: Pure Mathematics 33%, 2 hours, 100 marks	
Paper 3: Statistics and Mechanics 33%, 2 hours, 100 marks	Any A level applied content assessed Section A: Statistics (50 marks) Section B: Mechanics (50 marks)

A level Further Mathematics – Exam Board Edexcel	
Paper 1: Core Pure Mathematics 1 25%, 1 hour 30 seconds, 75 marks	Any A level core pure mathematics content assessed
Paper 2: Core Pure Mathematics 2 25%, 1 hour 30 seconds, 75 marks	
Paper 3: Further Mathematics Option 1 25%, 1 hour 30 seconds, 75 marks	Students take two optional papers with options available in: • Further Mechanics • Decision Mathematics
Paper 4: Further Mathematics Option 2 25%, 1 hour 30 seconds, 75 marks	

Equipment needed- A4 ring binder, dividers, lined or squared paper, pencil, pens, rulers and **A SCIENTIFIC CALCULATOR – FX991CW Casio (recommended). FX991EX** is also acceptable, earlier models won't have the required functions .

For any questions please contact
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Expanding brackets and simplifying expressions

Key points

- When you expand one set of brackets you must multiply everything inside the bracket by what is outside.
- When you expand two linear expressions, each with two terms of the form $ax + b$, where $a \neq 0$ and $b \neq 0$, you create four terms. Two of these can usually be simplified by collecting like terms.

Examples

Example 1   Expand $4(3x - 2)$

$4(3x - 2) = 12x - 8$	Multiply everything inside the bracket by the 4 outside the bracket

Example 2   Expand and simplify $3(x + 5) - 4(2x + 3)$

$3(x + 5) - 4(2x + 3)$ $= 3x + 15 - 8x - 12$ $= 3 - 5x$	**1** Expand each set of brackets separately by multiplying $(x + 5)$ by 3 and $(2x + 3)$ by -4 **2** Simplify by collecting like terms: $3x - 8x = -5x$ and $15 - 12 = 3$

Example 3   Expand and simplify $(x + 3)(x + 2)$

$(x + 3)(x + 2)$ $= x(x + 2) + 3(x + 2)$ $= x^2 + 2x + 3x + 6$ $= x^2 + 5x + 6$	**1** Expand the brackets by multiplying $(x + 2)$ by x and $(x + 2)$ by 3 **2** Simplify by collecting like terms: $2x + 3x = 5x$

Example 4   Expand and simplify $(x - 5)(2x + 3)$

$(x - 5)(2x + 3)$ $= x(2x + 3) - 5(2x + 3)$ $= 2x^2 + 3x - 10x - 15$ $= 2x^2 - 7x - 15$	**1** Expand the brackets by multiplying $(2x + 3)$ by x and $(2x + 3)$ by -5 **2** Simplify by collecting like terms: $3x - 10x = -7x$

Practice

1 Expand.

    a $3(2x - 1)$ **b** $-2(5pq + 4q^2)$

    c $-(3xy - 2y^2)$

2 Expand and simplify.

    a $7(3x + 5) + 6(2x - 8)$ **b** $8(5p - 2) - 3(4p + 9)$

    c $9(3s + 1) - 5(6s - 10)$ **d** $2(4x - 3) - (3x + 5)$

3 Expand.

    a $3x(4x + 8)$ **b** $4k(5k^2 - 12)$

    c $-2h(6h^2 + 11h - 5)$ **d** $-3s(4s^2 - 7s + 2)$

4 Expand and simplify.

    a $3(y^2 - 8) - 4(y^2 - 5)$ **b** $2x(x + 5) + 3x(x - 7)$

    c $4p(2p - 1) - 3p(5p - 2)$ **d** $3b(4b - 3) - b(6b - 9)$

5 Expand $\frac{1}{2}(2y - 8)$

6 Expand and simplify.

    a $13 - 2(m + 7)$ **b** $5p(p^2 + 6p) - 9p(2p - 3)$

7 The diagram shows a rectangle.

Write down an expression, in terms of x, for the area of the rectangle.

Show that the area of the rectangle can be written as $21x^2 - 35x$

$3x - 5$ [rectangle]

                             $7x$

8 Expand and simplify.

    a $(x + 4)(x + 5)$ **b** $(x + 7)(x + 3)$

    c $(x + 7)(x - 2)$ **d** $(x + 5)(x - 5)$

    e $(2x + 3)(x - 1)$ **f** $(3x - 2)(2x + 1)$

    g $(5x - 3)(2x - 5)$ **h** $(3x - 2)(7 + 4x)$

    i $(3x + 4y)(5y + 6x)$ **j** $(x + 5)^2$

    k $(2x - 7)^2$ **l** $(4x - 3y)^2$

Extend

9 Expand and simplify $(x + 3)^2 + (x - 4)^2$

10 Expand and simplify.

    a $\left(x + \dfrac{1}{x}\right)\left(x - \dfrac{2}{x}\right)$ **b** $\left(x + \dfrac{1}{x}\right)^2$

Pearson

Answers

1 **a** $6x - 3$ **b** $-10pq - 8q^2$
   c $-3xy + 2y^2$

2 **a** $21x + 35 + 12x - 48 = 33x - 13$
   b $40p - 16 - 12p - 27 = 28p - 43$
   c $27s + 9 - 30s + 50 = -3s + 59 = 59 - 3s$
   d $8x - 6 - 3x - 5 = 5x - 11$

3 **a** $12x^2 + 24x$ **b** $20k^3 - 48k$
   c $10h - 12h^3 - 22h^2$ **d** $21s^2 - 21s^3 - 6s$

4 **a** $-y^2 - 4$ **b** $5x^2 - 11x$
   c $2p - 7p^2$ **d** $6b^2$

5 $y - 4$

6 **a** $-1 - 2m$ **b** $5p^3 + 12p^2 + 27p$

7 $7x(3x - 5) = 21x^2 - 35x$

8 **a** $x^2 + 9x + 20$ **b** $x^2 + 10x + 21$
   c $x^2 + 5x - 14$ **d** $x^2 - 25$
   e $2x^2 + x - 3$ **f** $6x^2 - x - 2$
   g $10x^2 - 31x + 15$ **h** $12x^2 + 13x - 14$
   i $18x^2 + 39xy + 20y^2$ **j** $x^2 + 10x + 25$
   k $4x^2 - 28x + 49$ **l** $16x^2 - 24xy + 9y^2$

9 $2x^2 - 2x + 25$

10 **a** $x^2 - 1 - \dfrac{2}{x^2}$ **b** $x^2 + 2 + \dfrac{1}{x^2}$

Surds and rationalising the denominator

> **A LEVEL LINKS**
> Scheme of work: 1a. Algebraic expressions – basic algebraic manipulation, indices and surds

Key points

- A surd is the square root of a number that is not a square number, for example $\sqrt{2}, \sqrt{3}, \sqrt{5},$ etc.
- Surds can be used to give the exact value for an answer.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$
- To rationalise the denominator means to remove the surd from the denominator of a fraction.
- To rationalise $\dfrac{a}{\sqrt{b}}$ you multiply the numerator and denominator by the surd $\sqrt{b}$
- To rationalise $\dfrac{a}{b+\sqrt{c}}$ you multiply the numerator and denominator by $b-\sqrt{c}$

Examples

Example 1   Simplify $\sqrt{50}$

$\sqrt{50} = \sqrt{25 \times 2}$	1 Choose two numbers that are factors of 50. One of the factors must be a square number
$= \sqrt{25} \times \sqrt{2}$	2 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= 5 \times \sqrt{2}$	3 Use $\sqrt{25} = 5$
$= 5\sqrt{2}$	

Example 2   Simplify $\sqrt{147} - 2\sqrt{12}$

$\sqrt{147} - 2\sqrt{12}$ $= \sqrt{49 \times 3} - 2\sqrt{4 \times 3}$	1 Simplify $\sqrt{147}$ and $2\sqrt{12}$. Choose two numbers that are factors of 147 and two numbers that are factors of 12. One of each pair of factors must be a square number
$= \sqrt{49} \times \sqrt{3} - 2\sqrt{4} \times \sqrt{3}$	2 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= 7 \times \sqrt{3} - 2 \times 2 \times \sqrt{3}$	3 Use $\sqrt{49} = 7$ and $\sqrt{4} = 2$
$= 7\sqrt{3} - 4\sqrt{3}$	
$= 3\sqrt{3}$	4 Collect like terms

Example 3 Simplify $\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)$

$\left(\sqrt{7}+\sqrt{2}\right)\left(\sqrt{7}-\sqrt{2}\right)$ $= \sqrt{49} - \sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7} - \sqrt{4}$	**1** Expand the brackets. A common mistake here is to write $\left(\sqrt{7}\right)^2 = 49$
$= 7 - 2$ $= 5$	**2** Collect like terms: $-\sqrt{7}\sqrt{2} + \sqrt{2}\sqrt{7}$ $\qquad = -\sqrt{7}\sqrt{2} + \sqrt{7}\sqrt{2} = 0$

Example 4 Rationalise $\dfrac{1}{\sqrt{3}}$

$\dfrac{1}{\sqrt{3}} = \dfrac{1}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$	**1** Multiply the numerator and denominator by $\sqrt{3}$
$= \dfrac{1 \times \sqrt{3}}{\sqrt{9}}$	**2** Use $\sqrt{9} = 3$
$= \dfrac{\sqrt{3}}{3}$	

Example 5 Rationalise and simplify $\dfrac{\sqrt{2}}{\sqrt{12}}$

$\dfrac{\sqrt{2}}{\sqrt{12}} = \dfrac{\sqrt{2}}{\sqrt{12}} \times \dfrac{\sqrt{12}}{\sqrt{12}}$	**1** Multiply the numerator and denominator by $\sqrt{12}$
$= \dfrac{\sqrt{2} \times \sqrt{4 \times 3}}{12}$	**2** Simplify $\sqrt{12}$ in the numerator. Choose two numbers that are factors of 12. One of the factors must be a square number
	3 Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
$= \dfrac{2\sqrt{2}\sqrt{3}}{12}$	**4** Use $\sqrt{4} = 2$
	5 Simplify the fraction:
$= \dfrac{\sqrt{2}\sqrt{3}}{6}$	$\dfrac{2}{12}$ simplifies to $\dfrac{1}{6}$

Example 6 Rationalise and simplify $\dfrac{3}{2+\sqrt{5}}$

$\dfrac{3}{2+\sqrt{5}} = \dfrac{3}{2+\sqrt{5}} \times \dfrac{2-\sqrt{5}}{2-\sqrt{5}}$	**1** Multiply the numerator and denominator by $2-\sqrt{5}$
$= \dfrac{3\left(2-\sqrt{5}\right)}{\left(2+\sqrt{5}\right)\left(2-\sqrt{5}\right)}$	**2** Expand the brackets
$= \dfrac{6-3\sqrt{5}}{4+2\sqrt{5}-2\sqrt{5}-5}$	**3** Simplify the fraction
$= \dfrac{6-3\sqrt{5}}{-1}$	**4** Divide the numerator by -1 Remember to change the sign of all terms when dividing by -1
$= 3\sqrt{5}-6$	

Practice

1 Simplify.

  a $\sqrt{45}$ **b** $\sqrt{125}$

  c $\sqrt{48}$ **d** $\sqrt{175}$

  e $\sqrt{300}$ **f** $\sqrt{28}$

  g $\sqrt{72}$ **h** $\sqrt{162}$

> **Hint**
>
> One of the two numbers you choose at the start must be a square number.

2 Simplify.

  a $\sqrt{72}+\sqrt{162}$ **b** $\sqrt{45}-2\sqrt{5}$

  c $\sqrt{50}-\sqrt{8}$ **d** $\sqrt{75}-\sqrt{48}$

  e $2\sqrt{28}+\sqrt{28}$ **f** $2\sqrt{12}-\sqrt{12}+\sqrt{27}$

> **Watch out!**
>
> Check you have chosen the highest square number at the start.

3 Expand and simplify.

  a $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$ **b** $(3+\sqrt{3})(5-\sqrt{12})$

  c $(4-\sqrt{5})(\sqrt{45}+2)$ **d** $(5+\sqrt{2})(6-\sqrt{8})$

4   Rationalise and simplify, if possible.

    a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{1}{\sqrt{11}}$

    c $\dfrac{2}{\sqrt{7}}$ **d** $\dfrac{2}{\sqrt{8}}$

    e $\dfrac{2}{\sqrt{2}}$ **f** $\dfrac{5}{\sqrt{5}}$

    g $\dfrac{\sqrt{8}}{\sqrt{24}}$ **h** $\dfrac{\sqrt{5}}{\sqrt{45}}$

5   Rationalise and simplify.

    a $\dfrac{1}{3-\sqrt{5}}$ **b** $\dfrac{2}{4+\sqrt{3}}$ **c** $\dfrac{6}{5-\sqrt{2}}$

Extend

6   Expand and simplify $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)$

7   Rationalise and simplify, if possible.

    a $\dfrac{1}{\sqrt{9}-\sqrt{8}}$ **b** $\dfrac{1}{\sqrt{x}-\sqrt{y}}$

Answers

1   a $3\sqrt{5}$ b $5\sqrt{5}$

    c $4\sqrt{3}$ d $5\sqrt{7}$

    e $10\sqrt{3}$ f $2\sqrt{7}$

    g $6\sqrt{2}$ h $9\sqrt{2}$

2   a $15\sqrt{2}$ b $\sqrt{5}$

    c $3\sqrt{2}$ d $\sqrt{3}$

    e $6\sqrt{7}$ f $5\sqrt{3}$

3   a -1 b $9-\sqrt{3}$

    c $10\sqrt{5}-7$ d $26-4\sqrt{2}$

4   a $\dfrac{\sqrt{5}}{5}$ b $\dfrac{\sqrt{11}}{11}$

    c $\dfrac{2\sqrt{7}}{7}$ d $\dfrac{\sqrt{2}}{2}$

    e $\sqrt{2}$ f $\sqrt{5}$

    g $\dfrac{\sqrt{3}}{3}$ h $\dfrac{1}{3}$

5   a $\dfrac{3+\sqrt{5}}{4}$ b $\dfrac{2(4-\sqrt{3})}{13}$ c $\dfrac{6(5+\sqrt{2})}{23}$

6   $x-y$

7   a $3+2\sqrt{2}$ b $\dfrac{\sqrt{x}+\sqrt{y}}{x-y}$

Rules of indices

Key points

- $a^m \times a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^0 = 1$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$ i.e. the nth root of a
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $a^{-m} = \dfrac{1}{a^m}$
- The square root of a number produces two solutions, e.g. $\sqrt{16} = \pm 4$.

Examples

Example 1 Evaluate 10^0

$10^0 = 1$	Any value raised to the power of zero is equal to 1

Example 2 Evaluate $9^{\frac{1}{2}}$

$9^{\frac{1}{2}} = \sqrt{9}$ $\quad = 3$	Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$

Example 3 Evaluate $27^{\frac{2}{3}}$

$27^{\frac{2}{3}} = \left(\sqrt[3]{27}\right)^2$ $\quad = 3^2$ $\quad = 9$	**1** Use the rule $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ **2** Use $\sqrt[3]{27} = 3$

Example 4   Evaluate 4^{-2}

$4^{-2} = \dfrac{1}{4^2}$ $= \dfrac{1}{16}$	**1** Use the rule $a^{-m} = \dfrac{1}{a^m}$ **2** Use $4^2 = 16$

Example 5   Simplify $\dfrac{6x^5}{2x^2}$

$\dfrac{6x^5}{2x^2} = 3x^3$	$6 \div 2 = 3$ and use the rule $\dfrac{a^m}{a^n} = a^{m-n}$ to give $\dfrac{x^5}{x^2} = x^{5-2} = x^3$

Example 6   Simplify $\dfrac{x^3 \times x^5}{x^4}$

$\dfrac{x^3 \times x^5}{x^4} = \dfrac{x^{3+5}}{x^4} = \dfrac{x^8}{x^4}$ $= x^{8-4} = x^4$	**1** Use the rule $a^m \times a^n = a^{m+n}$ **2** Use the rule $\dfrac{a^m}{a^n} = a^{m-n}$

Example 7   Write $\dfrac{1}{3x}$ as a single power of x

$\dfrac{1}{3x} = \dfrac{1}{3}x^{-1}$	Use the rule $\dfrac{1}{a^m} = a^{-m}$, note that the fraction $\dfrac{1}{3}$ remains unchanged

Example 8   Write $\dfrac{4}{\sqrt{x}}$ as a single power of x

$\dfrac{4}{\sqrt{x}} = \dfrac{4}{x^{\frac{1}{2}}}$ $= 4x^{-\frac{1}{2}}$	**1** Use the rule $a^{\frac{1}{n}} = \sqrt[n]{a}$ **2** Use the rule $\dfrac{1}{a^m} = a^{-m}$

Practice

1 Evaluate.

    a 14^0 **b** 3^0 **c** 5^0 **d** x^0

2 Evaluate.

    a $49^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $125^{\frac{1}{3}}$ **d** $16^{\frac{1}{4}}$

3 Evaluate.

    a $25^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $49^{\frac{3}{2}}$ **d** $16^{\frac{3}{4}}$

4 Evaluate.

    a 5^{-2} **b** 4^{-3} **c** 2^{-5} **d** 6^{-2}

5 Simplify.

    a $\dfrac{3x^2 \times x^3}{2x^2}$ **b** $\dfrac{10x^5}{2x^2 \times x}$

    c $\dfrac{3x \times 2x^3}{2x^3}$ **d** $\dfrac{7x^3 y^2}{14x^5 y}$

    e $\dfrac{y^2}{y^{\frac{1}{2}} \times y}$ **f** $\dfrac{c^{\frac{1}{2}}}{c^2 \times c^{\frac{3}{2}}}$

    g $\dfrac{\left(2x^2\right)^3}{4x^0}$ **h** $\dfrac{x^{\frac{1}{2}} \times x^{\frac{3}{2}}}{x^{-2} \times x^3}$

> **Watch out!**
>
> Remember that any value raised to the power of zero is 1. This is the rule $a^0 = 1$.

6 Evaluate.

    a $4^{-\frac{1}{2}}$ **b** $27^{-\frac{2}{3}}$ **c** $9^{-\frac{1}{2}} \times 2^3$

    d $16^{\frac{1}{4}} \times 2^{-3}$ **e** $\left(\dfrac{9}{16}\right)^{-\frac{1}{2}}$ **f** $\left(\dfrac{27}{64}\right)^{-\frac{2}{3}}$

7 Write the following as a single power of x.

    a $\dfrac{1}{x}$ **b** $\dfrac{1}{x^7}$ **c** $\sqrt[4]{x}$

    d $\sqrt[5]{x^2}$ **e** $\dfrac{1}{\sqrt[3]{x}}$ **f** $\dfrac{1}{\sqrt[3]{x^2}}$

8 Write the following without negative or fractional powers.

    a x^{-3} **b** x^0 **c** $x^{\frac{1}{5}}$

    d $x^{\frac{2}{5}}$ **e** $x^{-\frac{1}{2}}$ **f** $x^{-\frac{3}{4}}$

9 Write the following in the form ax^n.

    a $5\sqrt{x}$ **b** $\dfrac{2}{x^3}$ **c** $\dfrac{1}{3x^4}$

    d $\dfrac{2}{\sqrt{x}}$ **e** $\dfrac{4}{\sqrt[3]{x}}$ **f** 3

Extend

10 Write as sums of powers of x.

    a $\dfrac{x^5+1}{x^2}$ **b** $x^2\left(x+\dfrac{1}{x}\right)$ **c** $x^{-4}\left(x^2+\dfrac{1}{x^3}\right)$

Answers

1 **a** 1 **b** 1 **c** 1 **d** 1

2 **a** 7 **b** 4 **c** 5 **d** 2

3 **a** 125 **b** 32 **c** 343 **d** 8

4 **a** $\dfrac{1}{25}$ **b** $\dfrac{1}{64}$ **c** $\dfrac{1}{32}$ **d** $\dfrac{1}{36}$

5 **a** $\dfrac{3x^3}{2}$ **b** $5x^2$

   c $3x$ **d** $\dfrac{y}{2x^2}$

   e $y^{\frac{1}{2}}$ **f** c^{-3}

   g $2x^6$ **h** x

6 **a** $\dfrac{1}{2}$ **b** $\dfrac{1}{9}$ **c** $\dfrac{8}{3}$

   d $\dfrac{1}{4}$ **e** $\dfrac{4}{3}$ **f** $\dfrac{16}{9}$

7 **a** x^{-1} **b** x^{-7} **c** $x^{\frac{1}{4}}$

   d $x^{\frac{2}{5}}$ **e** $x^{-\frac{1}{3}}$ **f** $x^{-\frac{2}{3}}$

8 **a** $\dfrac{1}{x^3}$ **b** 1 **c** $\sqrt[5]{x}$

   d $\sqrt[5]{x^2}$ **e** $\dfrac{1}{\sqrt{x}}$ **f** $\dfrac{1}{\sqrt[4]{x^3}}$

9 **a** $5x^{\frac{1}{2}}$ **b** $2x^{-3}$ **c** $\dfrac{1}{3}x^{-4}$

   d $2x^{-\frac{1}{2}}$ **e** $4x^{-\frac{1}{3}}$ **f** $3x^0$

10 **a** $x^3 + x^{-2}$ **b** $x^3 + x$ **c** $x^{-2} + x^{-7}$

Factorising expressions

Key points

- Factorising an expression is the opposite of expanding the brackets.
- A quadratic expression is in the form $ax^2 + bx + c$, where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose product is ac.
- An expression in the form $x^2 - y^2$ is called the difference of two squares. It factorises to $(x - y)(x + y)$.

Examples

Example 1   Factorise $15x^2y^3 + 9x^4y$

$15x^2y^3 + 9x^4y = 3x^2y(5y^2 + 3x^2)$	The highest common factor is $3x^2y$. So take $3x^2y$ outside the brackets and then divide each term by $3x^2y$ to find the terms in the brackets

Example 2   Factorise $4x^2 - 25y^2$

$4x^2 - 25y^2 = (2x + 5y)(2x - 5y)$	This is the difference of two squares as the two terms can be written as $(2x)^2$ and $(5y)^2$

Example 3   Factorise $x^2 + 3x - 10$

$b = 3,\ ac = -10$	1 Work out the two factors of $ac = -10$ which add to give $b = 3$ (5 and -2)
So $x^2 + 3x - 10 = x^2 + 5x - 2x - 10$	2 Rewrite the b term ($3x$) using these two factors
$= x(x + 5) - 2(x + 5)$	3 Factorise the first two terms and the last two terms
$= (x + 5)(x - 2)$	4 $(x + 5)$ is a factor of both terms

Example 4   Factorise $6x^2 - 11x - 10$

$b = -11$, $ac = -60$ So $6x^2 - 11x - 10 = 6x^2 - 15x + 4x - 10$ $= 3x(2x - 5) + 2(2x - 5)$ $= (2x - 5)(3x + 2)$	**1** Work out the two factors of $ac = -60$ which add to give $b = -11$ $(-15$ and $4)$ **2** Rewrite the b term $(-11x)$ using these two factors **3** Factorise the first two terms and the last two terms **4** $(2x - 5)$ is a factor of both terms

Example 5   Simplify $\dfrac{x^2 - 4x - 21}{2x^2 + 9x + 9}$

$\dfrac{x^2 - 4x - 21}{2x^2 + 9x + 9}$ For the numerator: $b = -4$, $ac = -21$ So $x^2 - 4x - 21 = x^2 - 7x + 3x - 21$ $= x(x - 7) + 3(x - 7)$ $= (x - 7)(x + 3)$ For the denominator: $b = 9$, $ac = 18$ So $2x^2 + 9x + 9 = 2x^2 + 6x + 3x + 9$ $= 2x(x + 3) + 3(x + 3)$ $= (x + 3)(2x + 3)$ So $\dfrac{x^2 - 4x - 21}{2x^2 + 9x + 9} = \dfrac{(x - 7)(x + 3)}{(x + 3)(2x + 3)}$ $= \dfrac{x - 7}{2x + 3}$	**1** Factorise the numerator and the denominator **2** Work out the two factors of $ac = -21$ which add to give $b = -4$ $(-7$ and $3)$ **3** Rewrite the b term $(-4x)$ using these two factors **4** Factorise the first two terms and the last two terms **5** $(x - 7)$ is a factor of both terms **6** Work out the two factors of $ac = 18$ which add to give $b = 9$ $(6$ and $3)$ **7** Rewrite the b term $(9x)$ using these two factors **8** Factorise the first two terms and the last two terms **9** $(x + 3)$ is a factor of both terms **10** $(x + 3)$ is a factor of both the numerator and denominator so cancels out as a value divided by itself is 1

Practice

1 Factorise.

   a $6x^4y^3 - 10x^3y^4$ **b** $21a^3b^5 + 35a^5b^2$

   c $25x^2y^2 - 10x^3y^2 + 15x^2y^3$

2 Factorise

   a $x^2 + 7x + 12$ **b** $x^2 + 5x - 14$

   c $x^2 - 11x + 30$ **d** $x^2 - 5x - 24$

   e $x^2 - 7x - 18$ **f** $x^2 + x - 20$

   g $x^2 - 3x - 40$ **h** $x^2 + 3x - 28$

3 Factorise

   a $36x^2 - 49y^2$ **b** $4x^2 - 81y^2$

   c $18a^2 - 200b^2c^2$

4 Factorise

   a $2x^2 + x - 3$ **b** $6x^2 + 17x + 5$

   c $2x^2 + 7x + 3$ **d** $9x^2 - 15x + 4$

   e $10x^2 + 21x + 9$ **f** $12x^2 - 38x + 20$

5 Simplify the algebraic fractions.

   a $\dfrac{2x^2 + 4x}{x^2 - x}$ **b** $\dfrac{x^2 + 3x}{x^2 + 2x - 3}$

   c $\dfrac{x^2 - 2x - 8}{x^2 - 4x}$ **d** $\dfrac{x^2 - 5x}{x^2 - 25}$

   e $\dfrac{x^2 - x - 12}{x^2 - 4x}$ **f** $\dfrac{2x^2 + 14x}{2x^2 + 4x - 70}$

6 Simplify

   a $\dfrac{9x^2 - 16}{3x^2 + 17x - 28}$ **b** $\dfrac{2x^2 - 7x - 15}{3x^2 - 17x + 10}$

   c $\dfrac{4 - 25x^2}{10x^2 - 11x - 6}$ **d** $\dfrac{6x^2 - x - 1}{2x^2 + 7x - 4}$

Extend

7 Simplify $\sqrt{x^2 + 10x + 25}$

8 Simplify $\dfrac{(x+2)^2 + 3(x+2)^2}{x^2 - 4}$

Answers

1 **a** $2x^3y^3(3x - 5y)$ **b** $7a^3b^2(3b^3 + 5a^2)$
 c $5x^2y^2(5 - 2x + 3y)$

2 **a** $(x + 3)(x + 4)$ **b** $(x + 7)(x - 2)$
 c $(x - 5)(x - 6)$ **d** $(x - 8)(x + 3)$
 e $(x - 9)(x + 2)$ **f** $(x + 5)(x - 4)$
 g $(x - 8)(x + 5)$ **h** $(x + 7)(x - 4)$

3 **a** $(6x - 7y)(6x + 7y)$ **b** $(2x - 9y)(2x + 9y)$
 c $2(3a - 10bc)(3a + 10bc)$

4 **a** $(x - 1)(2x + 3)$ **b** $(3x + 1)(2x + 5)$
 c $(2x + 1)(x + 3)$ **d** $(3x - 1)(3x - 4)$
 e $(5x + 3)(2x + 3)$ **f** $2(3x - 2)(2x - 5)$

5 **a** $\dfrac{2(x+2)}{x-1}$ **b** $\dfrac{x}{x-1}$

 c $\dfrac{x+2}{x}$ **d** $\dfrac{x}{x+5}$

 e $\dfrac{x+3}{x}$ **f** $\dfrac{x}{x-5}$

6 **a** $\dfrac{3x+4}{x+7}$ **b** $\dfrac{2x+3}{3x-2}$

 c $\dfrac{2-5x}{2x-3}$ **d** $\dfrac{3x+1}{x+4}$

7 $(x + 5)$

8 $\dfrac{4(x+2)}{x-2}$

Pearson

Completing the square

Key points

- Completing the square for a quadratic rearranges $ax^2 + bx + c$ into the form $p(x + q)^2 + r$
- If $a \neq 1$, then factorise using a as a common factor.

Examples

Example 1 Complete the square for the quadratic expression $x^2 + 6x - 2$

$x^2 + 6x - 2$	**1** Write $x^2 + bx + c$ in the form
$= (x + 3)^2 - 9 - 2$	$\left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2 + c$
$= (x + 3)^2 - 11$	**2** Simplify

Example 2 Write $2x^2 - 5x + 1$ in the form $p(x + q)^2 + r$

$2x^2 - 5x + 1$	**1** Before completing the square write $ax^2 + bx + c$ in the form $a\left(x^2 + \dfrac{b}{a}x\right) + c$
$= 2\left(x^2 - \dfrac{5}{2}x\right) + 1$	**2** Now complete the square by writing $x^2 - \dfrac{5}{2}x$ in the form $\left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$
$= 2\left[\left(x - \dfrac{5}{4}\right)^2 - \left(\dfrac{5}{4}\right)^2\right] + 1$	
$= 2\left(x - \dfrac{5}{4}\right)^2 - \dfrac{25}{8} + 1$	**3** Expand the square brackets – don't forget to multiply $\left(\dfrac{5}{4}\right)^2$ by the factor of 2
$= 2\left(x - \dfrac{5}{4}\right)^2 - \dfrac{17}{8}$	**4** Simplify

Practice

1   Write the following quadratic expressions in the form $(x + p)^2 + q$

    a $x^2 + 4x + 3$ **b** $x^2 - 10x - 3$

    c $x^2 - 8x$ **d** $x^2 + 6x$

    e $x^2 - 2x + 7$ **f** $x^2 + 3x - 2$

2   Write the following quadratic expressions in the form $p(x + q)^2 + r$

    a $2x^2 - 8x - 16$ **b** $4x^2 - 8x - 16$

    c $3x^2 + 12x - 9$ **d** $2x^2 + 6x - 8$

3   Complete the square.

    a $2x^2 + 3x + 6$ **b** $3x^2 - 2x$

    c $5x^2 + 3x$ **d** $3x^2 + 5x + 3$

Extend

4   Write $(25x^2 + 30x + 12)$ in the form $(ax + b)^2 + c$.

Pearson

Answers

1 a $(x + 2)^2 - 1$ b $(x - 5)^2 - 28$

   c $(x - 4)^2 - 16$ d $(x + 3)^2 - 9$

   e $(x - 1)^2 + 6$ f $\left(x + \dfrac{3}{2}\right)^2 - \dfrac{17}{4}$

2 a $2(x - 2)^2 - 24$ b $4(x - 1)^2 - 20$

   c $3(x + 2)^2 - 21$ d $2\left(x + \dfrac{3}{2}\right)^2 - \dfrac{25}{2}$

3 a $2\left(x + \dfrac{3}{4}\right)^2 + \dfrac{39}{8}$ b $3\left(x - \dfrac{1}{3}\right)^2 - \dfrac{1}{3}$

   c $5\left(x + \dfrac{3}{10}\right)^2 - \dfrac{9}{20}$ d $3\left(x + \dfrac{5}{6}\right)^2 + \dfrac{11}{12}$

4 $(5x + 3)^2 + 3$

Solving quadratic equations by factorisation

Key points

- A quadratic equation is an equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$.
- To factorise a quadratic equation find two numbers whose sum is b and whose products is ac.
- When the product of two numbers is 0, then at least one of the numbers must be 0.
- If a quadratic can be solved it will have two solutions (these may be equal).

Examples

Example 1   Solve $5x^2 = 15x$

$5x^2 = 15x$ $5x^2 - 15x = 0$ $5x(x - 3) = 0$ So $5x = 0$ or $(x - 3) = 0$ Therefore $x = 0$ or $x = 3$	**1** Rearrange the equation so that all of the terms are on one side of the equation and it is equal to zero. Do not divide both sides by x as this would lose the solution $x = 0$. **2** Factorise the quadratic equation. $5x$ is a common factor. **3** When two values multiply to make zero, at least one of the values must be zero. **4** Solve these two equations.

Example 2   Solve $x^2 + 7x + 12 = 0$

$x^2 + 7x + 12 = 0$ $b = 7, ac = 12$ $x^2 + 4x + 3x + 12 = 0$ $x(x + 4) + 3(x + 4) = 0$ $(x + 4)(x + 3) = 0$ So $(x + 4) = 0$ or $(x + 3) = 0$ Therefore $x = -4$ or $x = -3$	**1** Factorise the quadratic equation. Work out the two factors of $ac = 12$ which add to give you $b = 7$. (4 and 3) **2** Rewrite the b term ($7x$) using these two factors. **3** Factorise the first two terms and the last two terms. **4** $(x + 4)$ is a factor of both terms. **5** When two values multiply to make zero, at least one of the values must be zero. **6** Solve these two equations.

Example 3   Solve $9x^2 - 16 = 0$

$9x^2 - 16 = 0$ $(3x + 4)(3x - 4) = 0$ So $(3x + 4) = 0$ or $(3x - 4) = 0$ $x = -\dfrac{4}{3}$ or $x = \dfrac{4}{3}$	**1** Factorise the quadratic equation. This is the difference of two squares as the two terms are $(3x)^2$ and $(4)^2$. **2** When two values multiply to make zero, at least one of the values must be zero. **3** Solve these two equations.

Example 4   Solve $2x^2 - 5x - 12 = 0$

$b = -5$, $ac = -24$ So $2x^2 - 8x + 3x - 12 = 0$ $2x(x - 4) + 3(x - 4) = 0$ $(x - 4)(2x + 3) = 0$ So $(x - 4) = 0$ or $(2x + 3) = 0$ $x = 4$ or $x = -\dfrac{3}{2}$	**1** Factorise the quadratic equation. Work out the two factors of $ac = -24$ which add to give you $b = -5$. $(-8$ and $3)$ **2** Rewrite the b term $(-5x)$ using these two factors. **3** Factorise the first two terms and the last two terms. **4** $(x - 4)$ is a factor of both terms. **5** When two values multiply to make zero, at least one of the values must be zero. **6** Solve these two equations.

Practice

1 Solve

   a $6x^2 + 4x = 0$ **b** $28x^2 - 21x = 0$

   c $x^2 + 7x + 10 = 0$ **d** $x^2 - 5x + 6 = 0$

   e $x^2 - 3x - 4 = 0$ **f** $x^2 + 3x - 10 = 0$

   g $x^2 - 10x + 24 = 0$ **h** $x^2 - 36 = 0$

   i $x^2 + 3x - 28 = 0$ **j** $x^2 - 6x + 9 = 0$

   k $2x^2 - 7x - 4 = 0$ **l** $3x^2 - 13x - 10 = 0$

2 Solve

   a $x^2 - 3x = 10$ **b** $x^2 - 3 = 2x$

   c $x^2 + 5x = 24$ **d** $x^2 - 42 = x$

   e $x(x + 2) = 2x + 25$ **f** $x^2 - 30 = 3x - 2$

   g $x(3x + 1) = x^2 + 15$ **h** $3x(x - 1) = 2(x + 1)$

> **Hint**
> Get all terms onto one side of the equation.

Solving quadratic equations by completing the square

Key points

- Completing the square lets you write a quadratic equation in the form $p(x + q)^2 + r = 0$.

Examples

Example 5 Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$x^2 + 6x + 4 = 0$	**1** Write $x^2 + bx + c = 0$ in the form
$(x + 3)^2 - 9 + 4 = 0$	$\left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2 + c = 0$
$(x + 3)^2 - 5 = 0$	**2** Simplify.
$(x + 3)^2 = 5$	**3** Rearrange the equation to work out x. First, add 5 to both sides.
$x + 3 = \pm\sqrt{5}$	**4** Square root both sides. Remember that the square root of a value gives two answers.
$x = \pm\sqrt{5} - 3$	**5** Subtract 3 from both sides to solve the equation.
So $x = -\sqrt{5} - 3$ or $x = \sqrt{5} - 3$	**6** Write down both solutions.

Example 6 Solve $2x^2 - 7x + 4 = 0$. Give your solutions in surd form.

$2x^2 - 7x + 4 = 0$	**1** Before completing the square write $ax^2 + bx + c$ in the form
$2\left(x^2 - \dfrac{7}{2}x\right) + 4 = 0$	$a\left(x^2 + \dfrac{b}{a}x\right) + c$
$2\left[\left(x - \dfrac{7}{4}\right)^2 - \left(\dfrac{7}{4}\right)^2\right] + 4 = 0$	**2** Now complete the square by writing $x^2 - \dfrac{7}{2}x$ in the form $\left(x + \dfrac{b}{2a}\right)^2 - \left(\dfrac{b}{2a}\right)^2$
$2\left(x - \dfrac{7}{4}\right)^2 - \dfrac{49}{8} + 4 = 0$	**3** Expand the square brackets.
$2\left(x - \dfrac{7}{4}\right)^2 - \dfrac{17}{8} = 0$	**4** Simplify.

$2\left(x - \dfrac{7}{4}\right)^2 = \dfrac{17}{8}$	**5** Rearrange the equation to work out x. First, add $\dfrac{17}{8}$ to both sides.
$\left(x - \dfrac{7}{4}\right)^2 = \dfrac{17}{16}$	**6** Divide both sides by 2.
$x - \dfrac{7}{4} = \pm\dfrac{\sqrt{17}}{4}$	**7** Square root both sides. Remember that the square root of a value gives two answers.
$x = \pm\dfrac{\sqrt{17}}{4} + \dfrac{7}{4}$	**8** Add $\dfrac{7}{4}$ to both sides.
So $x = \dfrac{7}{4} - \dfrac{\sqrt{17}}{4}$ or $x = \dfrac{7}{4} + \dfrac{\sqrt{17}}{4}$	**9** Write down both the solutions.

Practice

3 Solve by completing the square.

    a $x^2 - 4x - 3 = 0$ **b** $x^2 - 10x + 4 = 0$

    c $x^2 + 8x - 5 = 0$ **d** $x^2 - 2x - 6 = 0$

    e $2x^2 + 8x - 5 = 0$ **f** $5x^2 + 3x - 4 = 0$

4 Solve by completing the square.

    a $(x - 4)(x + 2) = 5$

    b $2x^2 + 6x - 7 = 0$

    c $x^2 - 5x + 3 = 0$

> **Hint**
>
> Get all terms onto one side of the equation.

Solving quadratic equations by using the formula

Key points

- Any quadratic equation of the form $ax^2 + bx + c = 0$ can be solved using the formula
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- If $b^2 - 4ac$ is negative then the quadratic equation does not have any real solutions.
- It is useful to write down the formula before substituting the values for a, b and c.

Examples

Example 7   Solve $x^2 + 6x + 4 = 0$. Give your solutions in surd form.

$a = 1, b = 6, c = 4$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**1** Identify a, b and c and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$x = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$	**2** Substitute $a = 1$, $b = 6$, $c = 4$ into the formula.
$x = \dfrac{-6 \pm \sqrt{20}}{2}$	**3** Simplify. The denominator is 2, but this is only because $a = 1$. The denominator will not always be 2.
$x = \dfrac{-6 \pm 2\sqrt{5}}{2}$	**4** Simplify $\sqrt{20}$. $\sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$
$x = -3 \pm \sqrt{5}$	**5** Simplify by dividing numerator and denominator by 2.
So $x = -3 - \sqrt{5}$ or $x = \sqrt{5} - 3$	**6** Write down both the solutions.

Example 8   Solve $3x^2 - 7x - 2 = 0$. Give your solutions in surd form.

$a = 3$, $b = -7$, $c = -2$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**1** Identify a, b and c, making sure you get the signs right and write down the formula. Remember that $-b \pm \sqrt{b^2 - 4ac}$ is all over $2a$, not just part of it.
$x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(-2)}}{2(3)}$	**2** Substitute $a = 3$, $b = -7$, $c = -2$ into the formula.
$x = \dfrac{7 \pm \sqrt{73}}{6}$ So $x = \dfrac{7 - \sqrt{73}}{6}$ or $x = \dfrac{7 + \sqrt{73}}{6}$	**3** Simplify. The denominator is 6 when $a = 3$. A common mistake is to always write a denominator of 2. **4** Write down both the solutions.

Practice

5 Solve, giving your solutions in surd form.

   a $3x^2 + 6x + 2 = 0$ **b** $2x^2 - 4x - 7 = 0$

6 Solve the equation $x^2 - 7x + 2 = 0$

   Give your solutions in the form $\dfrac{a \pm \sqrt{b}}{c}$, where a, b and c are integers.

7 Solve $10x^2 + 3x + 3 = 5$

   Give your solution in surd form.

> **Hint**
>
> Get all terms onto one side of the equation.

Extend

8 Choose an appropriate method to solve each quadratic equation, giving your answer in surd form when necessary.

   a $4x(x - 1) = 3x - 2$

   b $10 = (x + 1)^2$

   c $x(3x - 1) = 10$

Answers

1 **a** $x = 0$ or $x = -\dfrac{2}{3}$ **b** $x = 0$ or $x = \dfrac{3}{4}$

    c $x = -5$ or $x = -2$ **d** $x = 2$ or $x = 3$

    e $x = -1$ or $x = 4$ **f** $x = -5$ or $x = 2$

    g $x = 4$ or $x = 6$ **h** $x = -6$ or $x = 6$

    i $x = -7$ or $x = 4$ **j** $x = 3$

    k $x = -\dfrac{1}{2}$ or $x = 4$ **l** $x = -\dfrac{2}{3}$ or $x = 5$

2 **a** $x = -2$ or $x = 5$ **b** $x = -1$ or $x = 3$

    c $x = -8$ or $x = 3$ **d** $x = -6$ or $x = 7$

    e $x = -5$ or $x = 5$ **f** $x = -4$ or $x = 7$

    g $x = -3$ or $x = 2\dfrac{1}{2}$ **h** $x = -\dfrac{1}{3}$ or $x = 2$

3 **a** $x = 2 + \sqrt{7}$ or $x = 2 - \sqrt{7}$ **b** $x = 5 + \sqrt{21}$ or $x = 5 - \sqrt{21}$

    c $x = -4 + \sqrt{21}$ or $x = -4 - \sqrt{21}$ **d** $x = 1 + \sqrt{7}$ or $x = 1 - \sqrt{7}$

    e $x = -2 + \sqrt{6.5}$ or $x = -2 - \sqrt{6.5}$ **f** $x = \dfrac{-3 + \sqrt{89}}{10}$ or $x = \dfrac{-3 - \sqrt{89}}{10}$

4 **a** $x = 1 + \sqrt{14}$ or $x = 1 - \sqrt{14}$ **b** $x = \dfrac{-3 + \sqrt{23}}{2}$ or $x = \dfrac{-3 - \sqrt{23}}{2}$

    c $x = \dfrac{5 + \sqrt{13}}{2}$ or $x = \dfrac{5 - \sqrt{13}}{2}$

5 **a** $x = -1 + \dfrac{\sqrt{3}}{3}$ or $x = -1 - \dfrac{\sqrt{3}}{3}$ **b** $x = 1 + \dfrac{3\sqrt{2}}{2}$ or $x = 1 - \dfrac{3\sqrt{2}}{2}$

6 $x = \dfrac{7 + \sqrt{41}}{2}$ or $x = \dfrac{7 - \sqrt{41}}{2}$

7 $x = \dfrac{-3 + \sqrt{89}}{20}$ or $x = \dfrac{-3 - \sqrt{89}}{20}$

8 **a** $x = \dfrac{7 + \sqrt{17}}{8}$ or $x = \dfrac{7 - \sqrt{17}}{8}$

    b $x = -1 + \sqrt{10}$ or $x = -1 - \sqrt{10}$

    c $x = -1\dfrac{2}{3}$ or $x = 2$

Sketching quadratic graphs
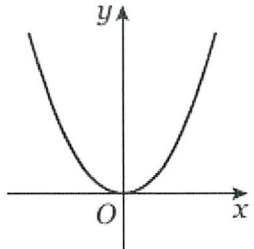
Key points

- The graph of the quadratic function $y = ax^2 + bx + c$, where $a \neq 0$, is a curve called a parabola.
- Parabolas have a line of symmetry and a shape as shown.

   for $a > 0$ for $a < 0$

- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- To find the coordinates of the maximum or minimum point (turning points) of a quadratic curve (parabola) you can use the completed square form of the function.

Examples

Example 1   Sketch the graph of $y = x^2$.

	The graph of $y = x^2$ is a parabola. When $x = 0$, $y = 0$. $a = 1$ which is greater than zero, so the graph has the shape:

Example 2   Sketch the graph of $y = x^2 - x - 6$.

When $x = 0$, $y = 0^2 - 0 - 6 = -6$ So the graph intersects the y-axis at $(0, -6)$ When $y = 0$, $x^2 - x - 6 = 0$ $(x + 2)(x - 3) = 0$ $x = -2$ or $x = 3$ So, the graph intersects the x-axis at $(-2, 0)$ and $(3, 0)$	**1** Find where the graph intersects the y-axis by substituting $x = 0$. **2** Find where the graph intersects the x-axis by substituting $y = 0$. **3** Solve the equation by factorising. **4** Solve $(x + 2) = 0$ and $(x - 3) = 0$. **5** $a = 1$ which is greater than zero, so the graph has the shape:

(continued on next page)

$$x^2 - x - 6 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 6$$

$$= \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}$$

When $\left(x - \frac{1}{2}\right)^2 = 0$, $x = \frac{1}{2}$ and

$y = -\frac{25}{4}$, so the turning point is at the

point $\left(\frac{1}{2}, -\frac{25}{4}\right)$
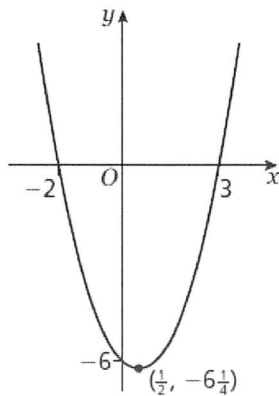
6 To find the turning point, complete the square.

7 The turning point is the minimum value for this expression and occurs when the term in the bracket is equal to zero.

Practice

1 Sketch the graph of $y = -x^2$.

2 Sketch each graph, labelling where the curve crosses the axes.
   a $y = (x + 2)(x - 1)$ b $y = x(x - 3)$ c $y = (x + 1)(x + 5)$

3 Sketch each graph, labelling where the curve crosses the axes.
   a $y = x^2 - x - 6$ b $y = x^2 - 5x + 4$ c $y = x^2 - 4$
   d $y = x^2 + 4x$ e $y = 9 - x^2$ f $y = x^2 + 2x - 3$

4 Sketch the graph of $y = 2x^2 + 5x - 3$, labelling where the curve crosses the axes.

Extend

5 Sketch each graph. Label where the curve crosses the axes and write down the coordinates of the turning point.
   a $y = x^2 - 5x + 6$ b $y = -x^2 + 7x - 12$ c $y = -x^2 + 4x$

6 Sketch the graph of $y = x^2 + 2x + 1$. Label where the curve crosses the axes and write down the equation of the line of symmetry.

Answers

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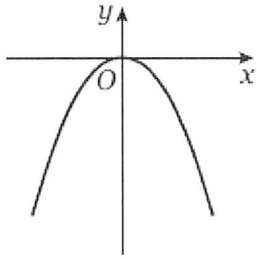
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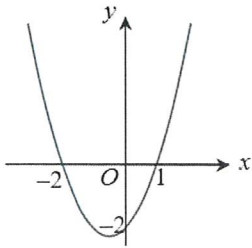
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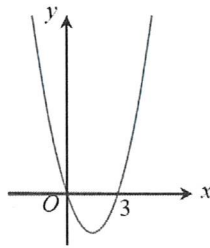
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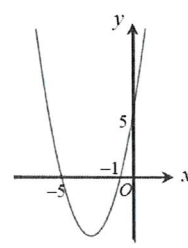
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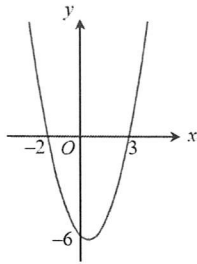
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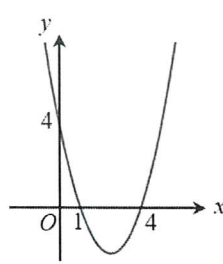
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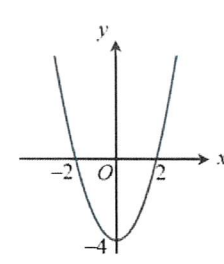
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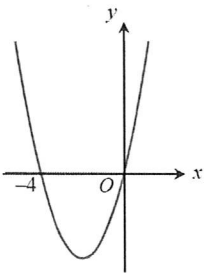
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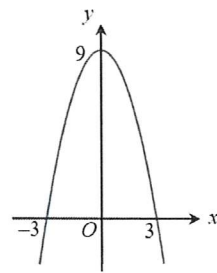
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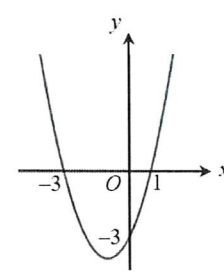
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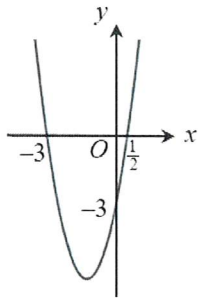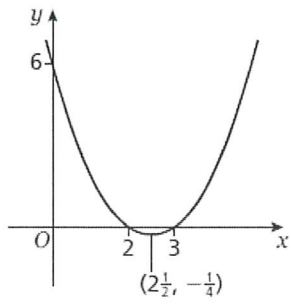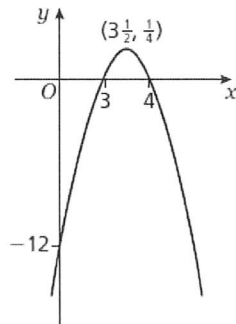
5 **a**   **b**   **c**
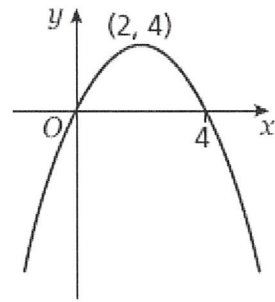
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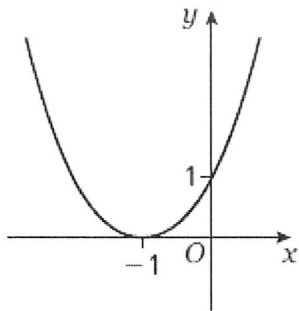
Line of symmetry at $x = -1$.

Solving linear simultaneous equations using the elimination method

A LEVEL LINKS
Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Two equations are simultaneous when they are both true at the same time.
- Solving simultaneous linear equations in two unknowns involves finding the value of each unknown which works for both equations.
- Make sure that the coefficient of one of the unknowns is the same in both equations.
- Eliminate this equal unknown by either subtracting or adding the two equations.

Examples

Example 1   Solve the simultaneous equations $3x + y = 5$ and $x + y = 1$

$3x + y = 5$ $-\quad x + y = 1$ $\overline{2x = 4}$ So $x = 2$	**1** Subtract the second equation from the first equation to eliminate the y term.
Using $x + y = 1$ $2 + y = 1$ So $y = -1$	**2** To find the value of y, substitute $x = 2$ into one of the original equations.

Check:
   equation 1: $3 \times 2 + (-1) = 5$ YES
   equation 2: $2 + (-1) = 1$ YES
	3 Substitute the values of x and y into both equations to check your answers.

Example 2   Solve $x + 2y = 13$ and $5x - 2y = 5$ simultaneously.

$x + 2y = 13$ $+\quad 5x - 2y = 5$ $\overline{6x = 18}$ So $x = 3$	**1** Add the two equations together to eliminate the y term.
Using $x + 2y = 13$ $3 + 2y = 13$ So $y = 5$	**2** To find the value of y, substitute $x = 3$ into one of the original equations.

Check:
   equation 1: $3 + 2 \times 5 = 13$ YES
   equation 2: $5 \times 3 - 2 \times 5 = 5$ YES
	3 Substitute the values of x and y into both equations to check your answers.

Example 3   Solve $2x + 3y = 2$ and $5x + 4y = 12$ simultaneously.

$(2x + 3y = 2) \times 4 \;\rightarrow$ $\qquad 8x + 12y = 8$ $(5x + 4y = 12) \times 3 \rightarrow$ $\underline{\qquad 15x + 12y = 36}$ $\qquad\qquad\qquad\qquad\; 7x \quad\;\; = 28$ So $x = 4$	**1** Multiply the first equation by 4 and the second equation by 3 to make the coefficient of y the same for both equations. Then subtract the first equation from the second equation to eliminate the y term.
Using $2x + 3y = 2$ $\qquad\quad 2 \times 4 + 3y = 2$ So $y = -2$	**2** To find the value of y, substitute $x = 4$ into one of the original equations.
Check: $\quad$ equation 1: $2 \times 4 + 3 \times (-2) = 2$ YES $\quad$ equation 2: $5 \times 4 + 4 \times (-2) = 12$ YES	**3** Substitute the values of x and y into both equations to check your answers.

Practice

Solve these simultaneous equations.

1 $4x + y = 8$
$\quad\; x + y = 5$

2 $3x + y = 7$
$\quad\; 3x + 2y = 5$

3 $4x + y = 3$
$\quad\; 3x - y = 11$

4 $3x + 4y = 7$
$\quad\; x - 4y = 5$

5 $2x + y = 11$
$\quad\; x - 3y = 9$

6 $2x + 3y = 11$
$\quad\; 3x + 2y = 4$

Solving linear simultaneous equations using the substitution method

Key points

- The subsitution method is the method most commonly used for A level. This is because it is the method used to solve linear and quadratic simultaneous equations.

Examples

Example 4 Solve the simultaneous equations $y = 2x + 1$ and $5x + 3y = 14$

$5x + 3(2x + 1) = 14$	**1** Substitute $2x + 1$ for y into the second equation.
$5x + 6x + 3 = 14$	**2** Expand the brackets and simplify.
$11x + 3 = 14$	
$11x = 11$	**3** Work out the value of x.
So $x = 1$	
Using $y = 2x + 1$	**4** To find the value of y, substitute
$y = 2 \times 1 + 1$	$x = 1$ into one of the original
So $y = 3$	equations.
Check:	**5** Substitute the values of x and y into
equation 1: $3 = 2 \times 1 + 1$ YES	both equations to check your
equation 2: $5 \times 1 + 3 \times 3 = 14$ YES	answers.

Example 5 Solve $2x - y = 16$ and $4x + 3y = -3$ simultaneously.

$y = 2x - 16$	**1** Rearrange the first equation.
$4x + 3(2x - 16) = -3$	**2** Substitute $2x - 16$ for y into the second equation.
$4x + 6x - 48 = -3$	**3** Expand the brackets and simplify.
$10x - 48 = -3$	
$10x = 45$	**4** Work out the value of x.
So $x = 4\frac{1}{2}$	
Using $y = 2x - 16$	**5** To find the value of y, substitute
$y = 2 \times 4\frac{1}{2} - 16$	$x = 4\frac{1}{2}$ into one of the original
So $y = -7$	equations.
Check:	**6** Substitute the values of x and y into
equation 1: $2 \times 4\frac{1}{2} - (-7) = 16$ YES	both equations to check your
equation 2: $4 \times 4\frac{1}{2} + 3 \times (-7) = -3$ YES	answers.

Practice

Solve these simultaneous equations.

7 $y = x - 4$
    $2x + 5y = 43$

8 $y = 2x - 3$
    $5x - 3y = 11$

9 $2y = 4x + 5$
    $9x + 5y = 22$

10 $2x = y - 2$
    $8x - 5y = -11$

11 $3x + 4y = 8$
    $2x - y = -13$

12 $3y = 4x - 7$
    $2y = 3x - 4$

13 $3x = y - 1$
    $2y - 2x = 3$

14 $3x + 2y + 1 = 0$
    $4y = 8 - x$

Extend

15 Solve the simultaneous equations $3x + 5y - 20 = 0$ and $2(x + y) = \dfrac{3(y - x)}{4}$.

Answers

1    $x = 1, y = 4$

2    $x = 3, y = -2$

3    $x = 2, y = -5$

4    $x = 3, y = -\frac{1}{2}$

5    $x = 6, y = -1$

6    $x = -2, y = 5$

7    $x = 9, y = 5$

8    $x = -2, y = -7$

9    $x = \frac{1}{2}, y = 3\frac{1}{2}$

10    $x = \frac{1}{2}, y = 3$

11    $x = -4, y = 5$

12    $x = -2, y = -5$

13    $x = \frac{1}{4}, y = 1\frac{3}{4}$

14    $x = -2, y = 2\frac{1}{2}$

15    $x = -2\frac{1}{2}, y = 5\frac{1}{2}$

Solving linear and quadratic simultaneous equations

A LEVEL LINKS
Scheme of work: 1c. Equations – quadratic/linear simultaneous

Key points

- Make one of the unknowns the subject of the linear equation (rearranging where necessary).
- Use the linear equation to substitute into the quadratic equation.
- There are usually two pairs of solutions.

Examples

Example 1   Solve the simultaneous equations $y = x + 1$ and $x^2 + y^2 = 13$

$x^2 + (x + 1)^2 = 13$	**1** Substitute $x + 1$ for y into the second equation.
$x^2 + x^2 + x + x + 1 = 13$ $2x^2 + 2x + 1 = 13$	**2** Expand the brackets and simplify.
$2x^2 + 2x - 12 = 0$ $(2x - 4)(x + 3) = 0$ So $x = 2$ or $x = -3$	**3** Factorise the quadratic equation. **4** Work out the values of x.
Using $y = x + 1$ When $x = 2$, $y = 2 + 1 = 3$ When $x = -3$, $y = -3 + 1 = -2$	**5** To find the value of y, substitute both values of x into one of the original equations.
So the solutions are $x = 2$, $y = 3$ and $x = -3$, $y = -2$	

Check:
   equation 1: $3 = 2 + 1$ YES
         and $-2 = -3 + 1$ YES
 	
   equation 2: $2^2 + 3^2 = 13$ YES
         and $(-3)^2 + (-2)^2 = 13$ YES
	6 Substitute both pairs of values of x and y into both equations to check your answers.

Example 2   Solve $2x + 3y = 5$ and $2y^2 + xy = 12$ simultaneously.

$x = \dfrac{5 - 3y}{2}$	**1** Rearrange the first equation.
$2y^2 + \left(\dfrac{5 - 3y}{2}\right)y = 12$	**2** Substitute $\dfrac{5 - 3y}{2}$ for x into the second equation. Notice how it is easier to substitute for x than for y.
$2y^2 + \dfrac{5y - 3y^2}{2} = 12$	**3** Expand the brackets and simplify.
$4y^2 + 5y - 3y^2 = 24$	
$y^2 + 5y - 24 = 0$	
$(y + 8)(y - 3) = 0$	**4** Factorise the quadratic equation.
So $y = -8$ or $y = 3$	**5** Work out the values of y.
Using $2x + 3y = 5$ When $y = -8$, $2x + 3 \times (-8) = 5$, $x = 14.5$ When $y = 3$, $2x + 3 \times 3 = 5$, $x = -2$	**6** To find the value of x, substitute both values of y into one of the original equations.
So the solutions are $x = 14.5,\ y = -8$ and $x = -2, y = 3$	

Check:
 equation 1: $2 \times 14.5 + 3 \times (-8) = 5$ YES
       and $2 \times (-2) + 3 \times 3 = 5$ YES
 equation 2: $2 \times (-8)^2 + 14.5 \times (-8) = 12$ YES
       and $2 \times (3)^2 + (-2) \times 3 = 12$ YES
	7 Substitute both pairs of values of x and y into both equations to check your answers.

Practice

Solve these simultaneous equations.

1 $y = 2x + 1$
    $x^2 + y^2 = 10$

2 $y = 6 - x$
    $x^2 + y^2 = 20$

3 $y = x - 3$
    $x^2 + y^2 = 5$

4 $y = 9 - 2x$
    $x^2 + y^2 = 17$

5 $y = 3x - 5$
    $y = x^2 - 2x + 1$

6 $y = x - 5$
    $y = x^2 - 5x - 12$

7 $y = x + 5$
    $x^2 + y^2 = 25$

8 $y = 2x - 1$
    $x^2 + xy = 24$

9 $y = 2x$
    $y^2 - xy = 8$

10 $2x + y = 11$
    $xy = 15$

Extend

11 $x - y = 1$
    $x^2 + y^2 = 3$

12 $y - x = 2$
    $x^2 + xy = 3$

Answers

1. $x = 1, y = 3$

   $x = -\dfrac{9}{5}, y = -\dfrac{13}{5}$

2. $x = 2, y = 4$

   $x = 4, y = 2$

3. $x = 1, y = -2$

   $x = 2, y = -1$

4. $x = 4, y = 1$

   $x = \dfrac{16}{5}, y = \dfrac{13}{5}$

5. $x = 3, y = 4$

   $x = 2, y = 1$

6. $x = 7, y = 2$

   $x = -1, y = -6$

7. $x = 0, y = 5$

   $x = -5, y = 0$

8. $x = -\dfrac{8}{3}, y = -\dfrac{19}{3}$

   $x = 3, y = 5$

9. $x = -2, y = -4$

   $x = 2, y = 4$

10. $x = \dfrac{5}{2}, y = 6$

    $x = 3, y = 5$

11. $x = \dfrac{1+\sqrt{5}}{2}, y = \dfrac{-1+\sqrt{5}}{2}$

    $x = \dfrac{1-\sqrt{5}}{2}, y = \dfrac{-1-\sqrt{5}}{2}$

12. $x = \dfrac{-1+\sqrt{7}}{2}, y = \dfrac{3+\sqrt{7}}{2}$

    $x = \dfrac{-1-\sqrt{7}}{2}, y = \dfrac{3-\sqrt{7}}{2}$

Solving simultaneous equations graphically
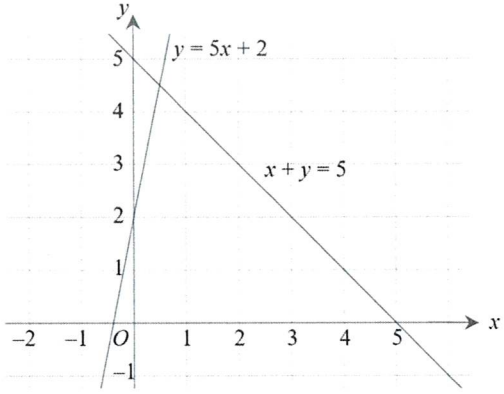
A LEVEL LINKS
Scheme of work: 1c. Equations – quadratic/linear simultaneous

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Key points

- You can solve any pair of simultaneous equations by drawing the graph of both equations and finding the point/points of intersection.

Examples

Example 1   Solve the simultaneous equations $y = 5x + 2$ and $x + y = 5$ graphically.

$y = 5 - x$	**1** Rearrange the equation $x + y = 5$ to make y the subject.
$y = 5 - x$ has gradient -1 and y-intercept 5. $y = 5x + 2$ has gradient 5 and y-intercept 2.	**2** Plot both graphs on the same grid using the gradients and y-intercepts.
Lines intersect at $x = 0.5, y = 4.5$ Check: First equation $y = 5x + 2$: $\quad 4.5 = 5 \times 0.5 + 2 \qquad$ YES Second equation $x + y = 5$: $\quad 0.5 + 4.5 = 5 \qquad$ YES	**3** The solutions of the simultaneous equations are the point of intersection. **4** Check your solutions by substituting the values into both equations.

Example 2   Solve the simultaneous equations $y = x - 4$ and $y = x^2 - 4x + 2$ graphically.

x	0	1	2	3	4
y	2	−1	−2	−1	2

1 Construct a table of values and calculate the points for the quadratic equation.

2 Plot the graph.

3 Plot the linear graph on the same grid using the gradient and y-intercept.
$y = x - 4$ has gradient 1 and y-intercept −4.
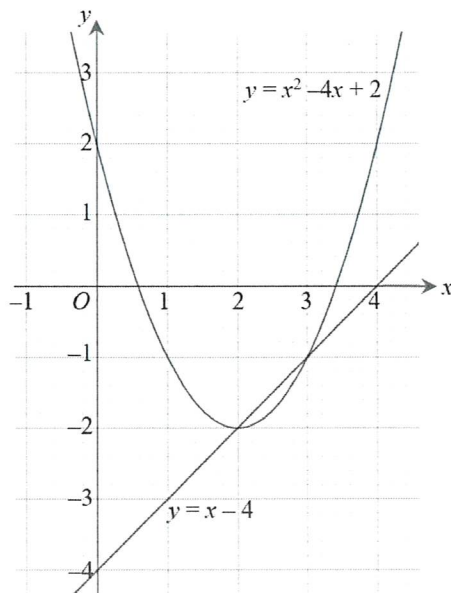
The line and curve intersect at
   $x = 3, y = -1$ and $x = 2, y = -2$

4 The solutions of the simultaneous equations are the points of intersection.

Check:
First equation $y = x - 4$:
   $-1 = 3 - 4$ YES
   $-2 = 2 - 4$ YES
Second equation $y = x^2 - 4x + 2$:
   $-1 = 3^2 - 4 \times 3 + 2$ YES
   $-2 = 2^2 - 4 \times 2 + 2$ YES

5 Check your solutions by substituting the values into both equations.

Practice

1   Solve these pairs of simultaneous equations graphically.
   a $y = 3x - 1$ and $y = x + 3$
   b $y = x - 5$ and $y = 7 - 5x$
   c $y = 3x + 4$ and $y = 2 - x$

2   Solve these pairs of simultaneous equations graphically.
   a $x + y = 0$ and $y = 2x + 6$
   b $4x + 2y = 3$ and $y = 3x - 1$
   c $2x + y + 4 = 0$ and $2y = 3x - 1$

Hint

Rearrange the equation to make y the subject.

3 Solve these pairs of simultaneous equations graphically.

    a $y = x - 1$ and $y = x^2 - 4x + 3$

    b $y = 1 - 3x$ and $y = x^2 - 3x - 3$

    c $y = 3 - x$ and $y = x^2 + 2x + 5$

4 Solve the simultaneous equations $x + y = 1$ and $x^2 + y^2 = 25$ graphically.

Extend

5   **a**   Solve the simultaneous equations $2x + y = 3$ and $x^2 + y = 4$

        i   graphically

        ii   algebraically to 2 decimal places.

    b   Which method gives the more accurate solutions? Explain your answer.

Answers

1   a $x = 2, y = 5$

     b $x = 2, y = -3$

     c $x = -0.5, y = 2.5$

2   a $x = -2, y = 2$

     b $x = 0.5, y = 0.5$

     c $x = -1, y = -2$

3   a $x = 1, y = 0$ and $x = 4, y = 3$

     b $x = -2, y = 7$ and $x = 2, y = -5$

     c $x = -2, y = 5$ and $x = -1, y = 4$

4    $x = -3, y = 4$ and $x = 4, y = -3$

5   a   i $x = 2.5, y = -2$ and $x = -0.5, y = 4$

         ii $x = 2.41, y = -1.83$ and $x = -0.41, y = 3.83$

     b   Solving algebraically gives the more accurate solutions as the solutions from the graph are only estimates, based on the accuracy of your graph.

Linear inequalities

Key points

- Solving linear inequalities uses similar methods to those for solving linear equations.
- When you multiply or divide an inequality by a negative number you need to reverse the inequality sign, e.g. < becomes >.

Examples

Example 1   Solve $-8 \leq 4x < 16$

$-8 \leq 4x < 16$ $-2 \leq x < 4$	Divide all three terms by 4.

Example 2   Solve $4 \leq 5x < 10$

$4 \leq 5x < 10$ $\frac{4}{5} \leq x < 2$	Divide all three terms by 5.

Example 3   Solve $2x - 5 < 7$

$2x - 5 < 7$ $2x < 12$ $x < 6$	**1** Add 5 to both sides. **2** Divide both sides by 2.

Example 4   Solve $2 - 5x \geq -8$

$2 - 5x \geq -8$ $-5x \geq -10$ $x \leq 2$	**1** Subtract 2 from both sides. **2** Divide both sides by -5. Remember to reverse the inequality when dividing by a negative number.

Example 5   Solve $4(x - 2) > 3(9 - x)$

$4(x - 2) > 3(9 - x)$ $4x - 8 > 27 - 3x$ $7x - 8 > 27$ $7x > 35$ $x > 5$	**1** Expand the brackets. **2** Add $3x$ to both sides. **3** Add 8 to both sides. **4** Divide both sides by 7.

Practice

1 Solve these inequalities.

    a $4x > 16$ **b** $5x - 7 \leq 3$ **c** $1 \geq 3x + 4$

    d $5 - 2x < 12$ **e** $\dfrac{x}{2} \geq 5$ **f** $8 < 3 - \dfrac{x}{3}$

2 Solve these inequalities.

    a $\dfrac{x}{5} < -4$ **b** $10 \geq 2x + 3$ **c** $7 - 3x > -5$

3 Solve

    a $2 - 4x \geq 18$ **b** $3 \leq 7x + 10 < 45$ **c** $6 - 2x \geq 4$

    d $4x + 17 < 2 - x$ **e** $4 - 5x < -3x$ **f** $-4x \geq 24$

4 Solve these inequalities.

    a $3t + 1 < t + 6$ **b** $2(3n - 1) \geq n + 5$

5 Solve.

    a $3(2 - x) > 2(4 - x) + 4$ **b** $5(4 - x) > 3(5 - x) + 2$

Extend

6 Find the set of values of x for which $2x + 1 > 11$ and $4x - 2 > 16 - 2x$.

Answers

1 **a** $x > 4$ **b** $x \leq 2$ **c** $x \leq -1$

   d $x > -\dfrac{7}{2}$ **e** $x \geq 10$ **f** $x < -15$

2 **a** $x < -20$ **b** $x \leq 3.5$ **c** $x < 4$

3 **a** $x \leq -4$ **b** $-1 \leq x < 5$ **c** $x \leq 1$

   d $x < -3$ **e** $x > 2$ **f** $x \leq -6$

4 **a** $t < \dfrac{5}{2}$ **b** $n \geq \dfrac{7}{5}$

5 **a** $x < -6$ **b** $x < \dfrac{3}{2}$

6 $x > 5$ (which also satisfies $x > 3$)
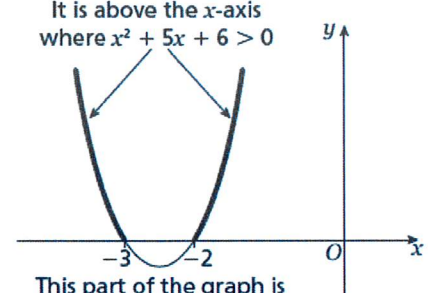
Quadratic inequalities

Key points
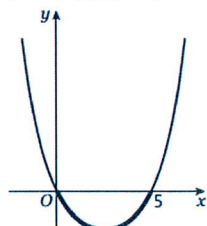
- First replace the inequality sign by = and solve the quadratic equation.
- Sketch the graph of the quadratic function.
- Use the graph to find the values which satisfy the quadratic inequality.

Examples
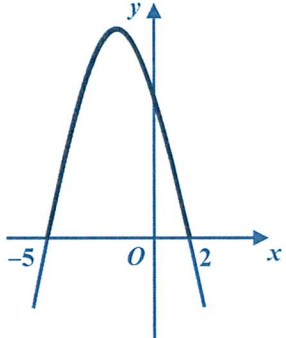
Example 1   Find the set of values of x which satisfy $x^2 + 5x + 6 > 0$

$x^2 + 5x + 6 = 0$ $(x + 3)(x + 2) = 0$ $x = -3$ or $x = -2$	**1** Solve the quadratic equation by factorising.
It is above the x-axis where $x^2 + 5x + 6 > 0$ This part of the graph is not needed as this is where $x^2 + 5x + 6 < 0$	**2** Sketch the graph of $y = (x + 3)(x + 2)$ **3** Identify on the graph where $x^2 + 5x + 6 > 0$, i.e. where $y > 0$
$x < -3$ or $x > -2$	**4** Write down the values which satisfy the inequality $x^2 + 5x + 6 > 0$

Example 2   Find the set of values of x which satisfy $x^2 - 5x \leq 0$

$x^2 - 5x = 0$ $x(x - 5) = 0$ $x = 0$ or $x = 5$ 	**1** Solve the quadratic equation by factorising. **2** Sketch the graph of $y = x(x - 5)$ **3** Identify on the graph where $x^2 - 5x \leq 0$, i.e. where $y \leq 0$
$0 \leq x \leq 5$	**4** Write down the values which satisfy the inequality $x^2 - 5x \leq 0$

Example 3   Find the set of values of x which satisfy $-x^2 - 3x + 10 \geq 0$

$-x^2 - 3x + 10 = 0$ $(-x + 2)(x + 5) = 0$ $x = 2$ or $x = -5$	**1** Solve the quadratic equation by factorising.
	2 Sketch the graph of $y = (-x + 2)(x + 5) = 0$ **3** Identify on the graph where $-x^2 - 3x + 10 \geq 0$, i.e. where $y \geq 0$
$-5 \leq x \leq 2$	**3** Write down the values which satisfy the inequality $-x^2 - 3x + 10 \geq 0$

Practice

1   Find the set of values of x for which $(x + 7)(x - 4) \leq 0$

2   Find the set of values of x for which $x^2 - 4x - 12 \geq 0$

3   Find the set of values of x for which $2x^2 - 7x + 3 < 0$

4   Find the set of values of x for which $4x^2 + 4x - 3 > 0$

5   Find the set of values of x for which $12 + x - x^2 \geq 0$

Extend

Find the set of values which satisfy the following inequalities.

6 $x^2 + x \leq 6$

7 $x(2x - 9) < -10$

8 $6x^2 \geq 15 + x$

Answers

1 $-7 \le x \le 4$

2 $x \le -2$ or $x \ge 6$

3 $\dfrac{1}{2} < x < 3$

4 $x < -\dfrac{3}{2}$ or $x > \dfrac{1}{2}$

5 $-3 \le x \le 4$

6 $-3 \le x \le 2$

7 $2 < x < 2\dfrac{1}{2}$

8 $x \le -\dfrac{3}{2}$ or $x \ge \dfrac{5}{3}$

Sketching cubic and reciprocal graphs

A LEVEL LINKS
Scheme of work: 1e. Graphs – cubic, quartic and reciprocal

Key points

- The graph of a cubic function, which can be written in the form $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$, has one of the shapes shown here.

for $a > 0$ for $a < 0$

$y = x^3$ $y = -x^3$
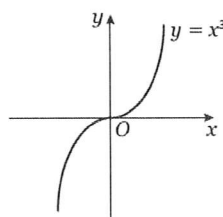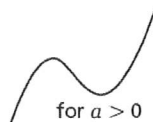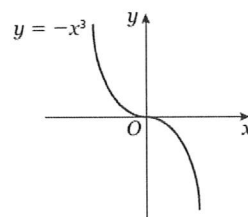
special case: $a = 1$ special case: $a = -1$
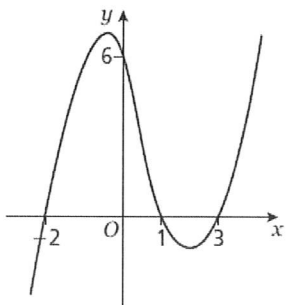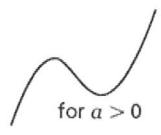
- The graph of a reciprocal function of the form $y = \dfrac{a}{x}$ has one of the shapes shown here.
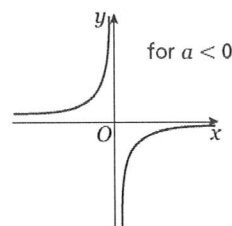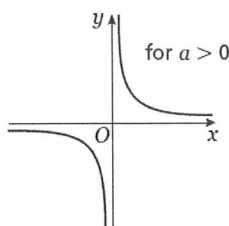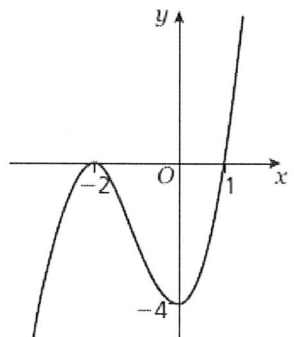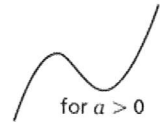
for $a > 0$ for $a < 0$

- To sketch the graph of a function, find the points where the graph intersects the axes.
- To find where the curve intersects the y-axis substitute $x = 0$ into the function.
- To find where the curve intersects the x-axis substitute $y = 0$ into the function.
- Where appropriate, mark and label the asymptotes on the graph.
- Asymptotes are lines (usually horizontal or vertical) which the curve gets closer to but never touches or crosses. Asymptotes usually occur with reciprocal functions. For example, the asymptotes for the graph of $y = \dfrac{a}{x}$ are the two axes (the lines $y = 0$ and $x = 0$).
- At the turning points of a graph the gradient of the curve is 0 and any tangents to the curve at these points are horizontal.
- A double root is when two of the solutions are equal. For example $(x - 3)^2(x + 2)$ has a double root at $x = 3$.
- When there is a double root, this is one of the turning points of a cubic function.

Examples

Example 1   Sketch the graph of $y = (x - 3)(x - 1)(x + 2)$

> To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.

When $x = 0$, $y = (0 - 3)(0 - 1)(0 + 2)$ $= (-3) \times (-1) \times 2 = 6$ The graph intersects the y-axis at $(0, 6)$ When $y = 0$, $(x - 3)(x - 1)(x + 2) = 0$ So $x = 3$, $x = 1$ or $x = -2$ The graph intersects the x-axis at $(-2, 0)$, $(1, 0)$ and $(3, 0)$ 	**1** Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. Make sure you get the coordinates the right way around, (x, y). **2** Solve the equation by solving $x - 3 = 0$, $x - 1 = 0$ and $x + 2 = 0$ **3** Sketch the graph. $a = 1 > 0$ so the graph has the shape: for $a > 0$

Example 2   Sketch the graph of $y = (x + 2)^2(x - 1)$

> To sketch a cubic curve find intersects with both axes and use the key points above for the correct shape.

When $x = 0$, $y = (0 + 2)^2(0 - 1)$ $= 2^2 \times (-1) = -4$ The graph intersects the y-axis at $(0, -4)$ When $y = 0$, $(x + 2)^2(x - 1) = 0$ So $x = -2$ or $x = 1$ $(-2, 0)$ is a turning point as $x = -2$ is a double root. The graph crosses the x-axis at $(1, 0)$ 	**1** Find where the graph intersects the axes by substituting $x = 0$ and $y = 0$. **2** Solve the equation by solving $x + 2 = 0$ and $x - 1 = 0$ **3** $a = 1 > 0$ so the graph has the shape: for $a > 0$

Practice

1 Here are six equations.

A $y = \dfrac{5}{x}$ **B** $y = x^2 + 3x - 10$ **C** $y = x^3 + 3x^2$

D $y = 1 - 3x^2 - x^3$ **E** $y = x^3 - 3x^2 - 1$ **F** $x + y = 5$

Hint

Find where each of the cubic equations cross the y-axis.

Here are six graphs.

i   **ii**   **iii**

iv   **v**   **vi**
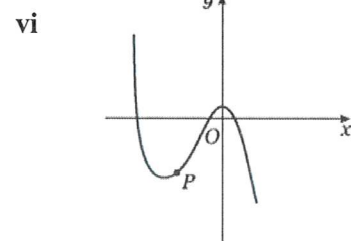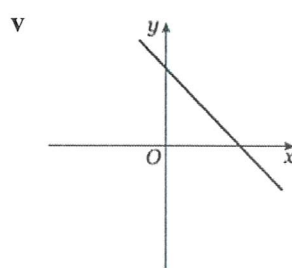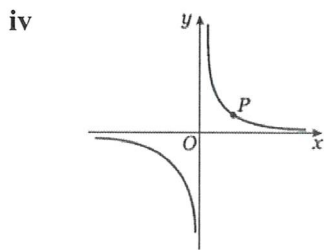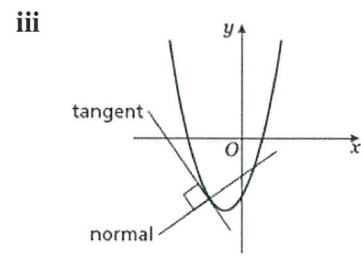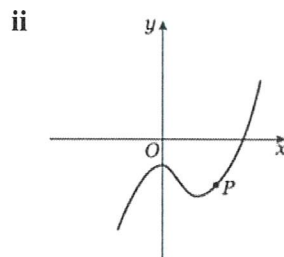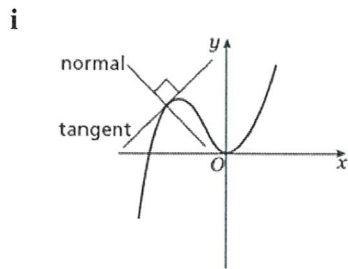
a Match each graph to its equation.

b Copy the graphs ii, iv and vi and draw the tangent and normal each at point P.

Sketch the following graphs

2 $y = 2x^3$

3 $y = x(x - 2)(x + 2)$

4 $y = (x + 1)(x + 4)(x - 3)$

5 $y = (x + 1)(x - 2)(1 - x)$

6 $y = (x - 3)^2(x + 1)$

7 $y = (x - 1)^2(x - 2)$

8 $y = \dfrac{3}{x}$

Hint: Look at the shape of $y = \dfrac{a}{x}$ in the second key point.

9 $y = -\dfrac{2}{x}$

Extend

10 Sketch the graph of $y = \dfrac{1}{x + 2}$

11 Sketch the graph of $y = \dfrac{1}{x - 1}$

Answers

1   a   i – C
        ii – E
        iii – B
        iv – A
        v – F
        vi – D
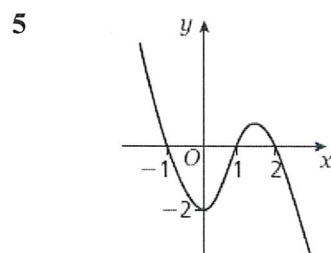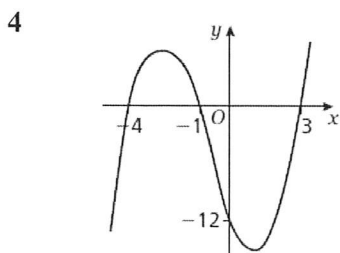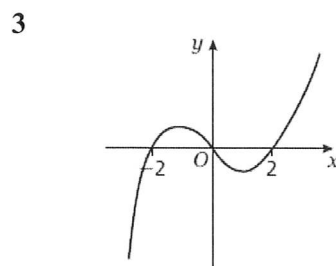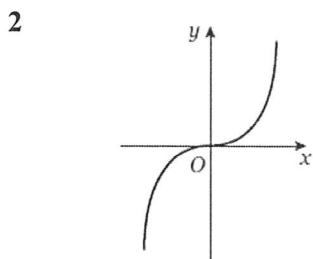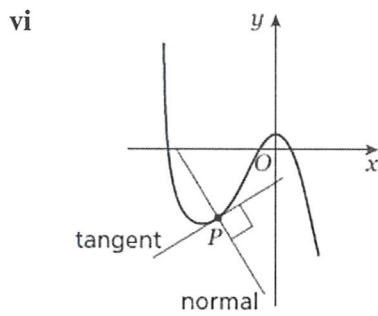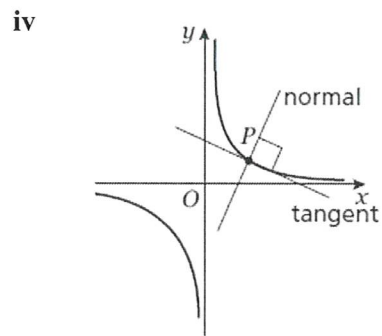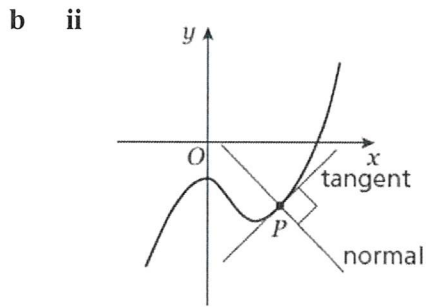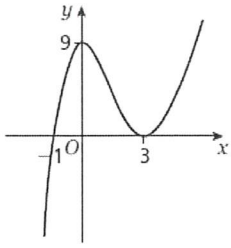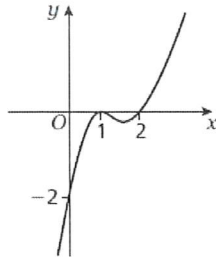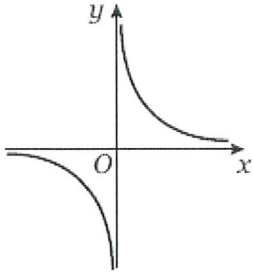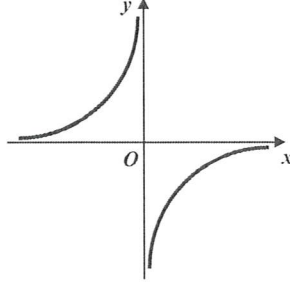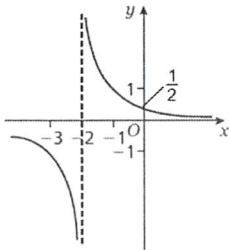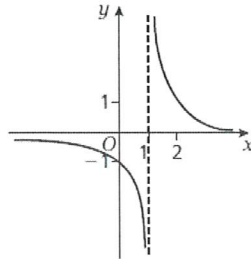
   b   **ii**

       iv

       vi

2

3

4

5

6

9

−1 O 3 x

7

O 1 2 x

−2

8

O x

9

O x

10

1 $\frac{1}{2}$

−3 −2 −1 O x

−1

11

1

O 1 2 x

−1

Translating graphs

Key points

- The transformation $y = f(x) \pm a$ is a translation of $y = f(x)$ parallel to the y-axis; it is a vertical translation.

  As shown on the graph,
  - $y = f(x) + a$ translates $y = f(x)$ up
  - $y = f(x) - a$ translates $y = f(x)$ down.
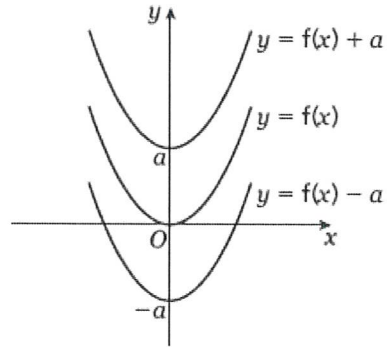
- The transformation $y = f(x \pm a)$ is a translation of $y = f(x)$ parallel to the x-axis; it is a horizontal translation.

  As shown on the graph,
  - $y = f(x + a)$ translates $y = f(x)$ to the left
  - $y = f(x - a)$ translates $y = f(x)$ to the right.
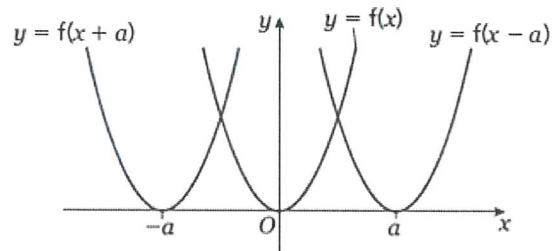
Examples

Example 1   The graph shows the function $y = f(x)$.

Sketch the graph of $y = f(x) + 2$.
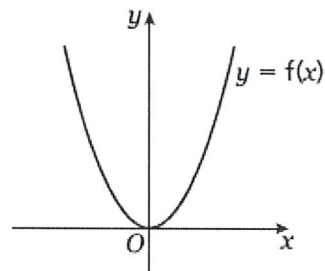
For the function $y = f(x) + 2$ translate the function $y = f(x)$ 2 units up.

Pearson

Example 2   The graph shows the function $y = f(x)$.

Sketch the graph of $y = f(x - 3)$.
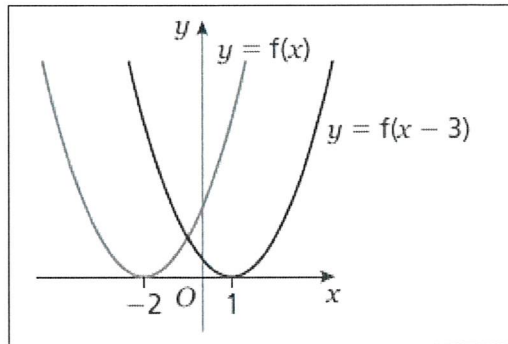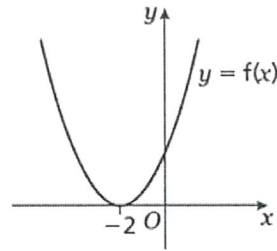
	For the function $y = f(x - 3)$ translate the function $y = f(x)$ 3 units right.

Practice

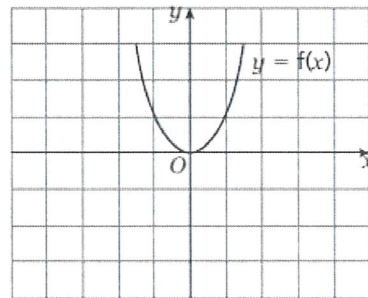**1**   The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch and
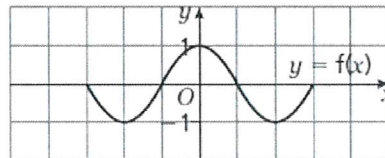label the graphs of $y = f(x) + 4$ and $y = f(x + 2)$.
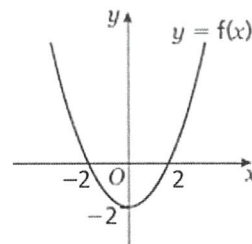
2   The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch and
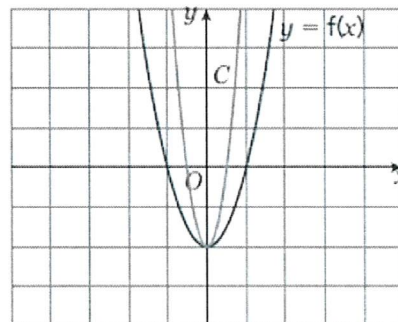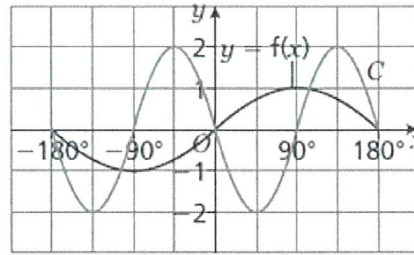label the graphs of $y = f(x + 3)$ and $y = f(x) - 3$.
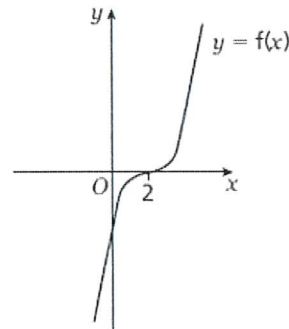
3   The graph shows the function $y = f(x)$.
Copy the graph and on the same axes sketch the
graph of $y = f(x - 5)$.
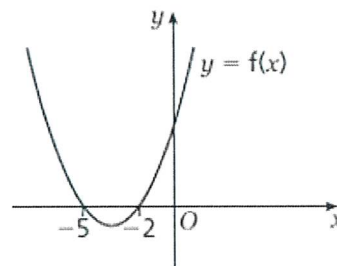
4   The graph shows the function $y = f(x)$ and two
    transformations of $y = f(x)$, labelled C_1 and C_2.
    Write down the equations of the translated curves
    C_1 and C_2 in function form.
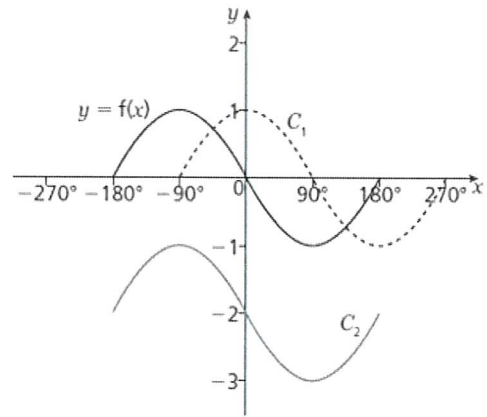
5   The graph shows the function $y = f(x)$ and two
    transformations of $y = f(x)$, labelled C_1 and C_2.
    Write down the equations of the translated curves
    C_1 and C_2 in function form.
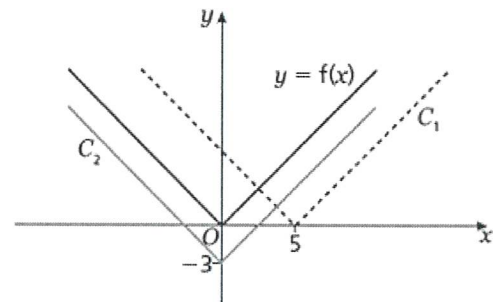
6   The graph shows the function $y = f(x)$.
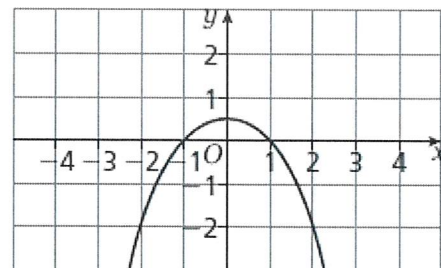
   a   Sketch the graph of $y = f(x) + 2$
    b   Sketch the graph of $y = f(x + 2)$

Stretching graphs

Key points

- The transformation $y = f(ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\dfrac{1}{a}$ parallel to the x-axis.
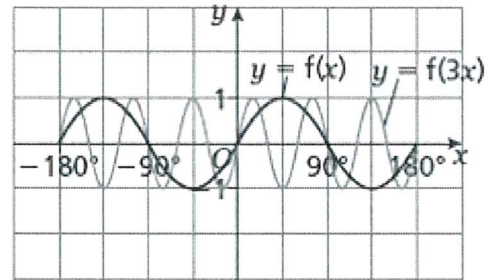
- The transformation $y = f(-ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\dfrac{1}{a}$ parallel to the x-axis and then a reflection in the y-axis.
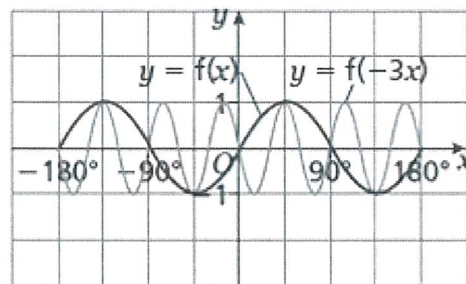
- The transformation $y = af(x)$ is a vertical stretch of $y = f(x)$ with scale factor a parallel to the y-axis.
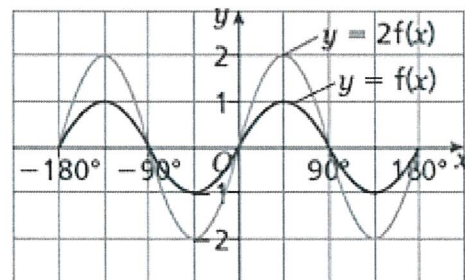
- The transformation $y = -af(x)$ is a vertical stretch of $y = f(x)$ with scale factor a parallel to the y-axis and then a reflection in the x-axis.
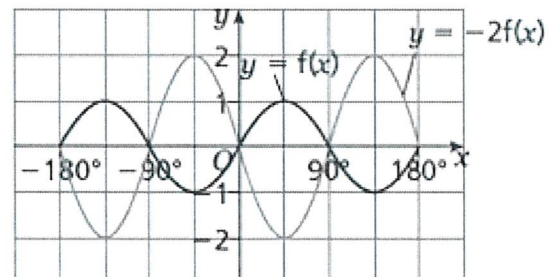
Examples

Example 3   The graph shows the function $y = f(x)$.

Sketch and label the graphs of
$y = 2f(x)$ and $y = -f(x)$.
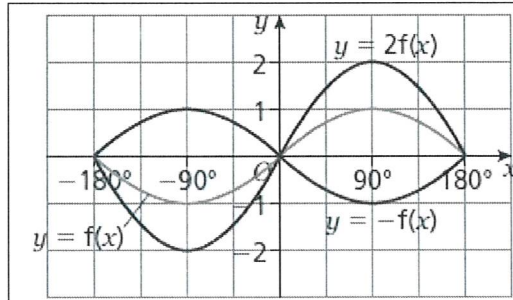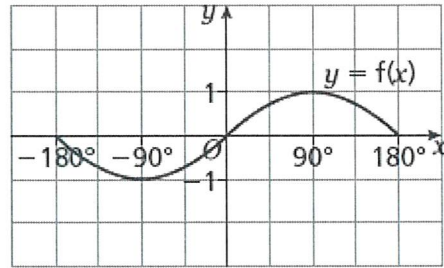
	The function $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2 parallel to the y-axis.
	The function $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis.

Example 4   The graph shows the function $y = f(x)$.

Sketch and label the graphs of
$y = f(2x)$ and $y = f(-x)$.
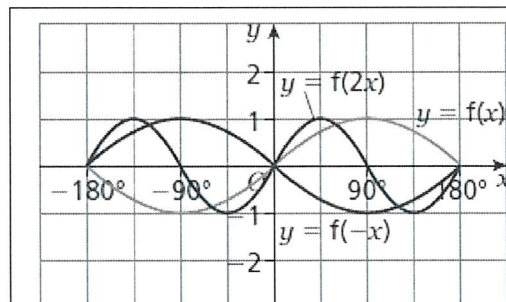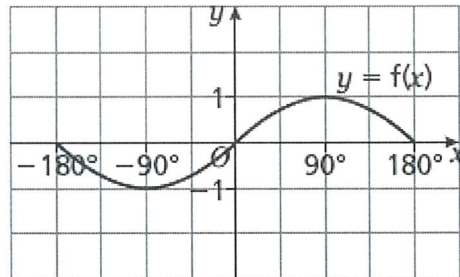
	The function $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$ parallel to the x-axis.
	The function $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.

Practice

7 The graph shows the function $y = f(x)$.

    a Copy the graph and on the same axes sketch and label the graph of $y = 3f(x)$.

    b Make another copy of the graph and on the same axes sketch and label the graph of $y = f(2x)$.
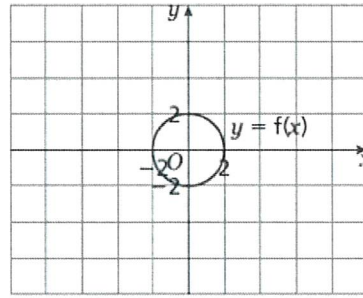
8 The graph shows the function $y = f(x)$. Copy the graph and on the same axes sketch and label the graphs of $y = -2f(x)$ and $y = f(3x)$.
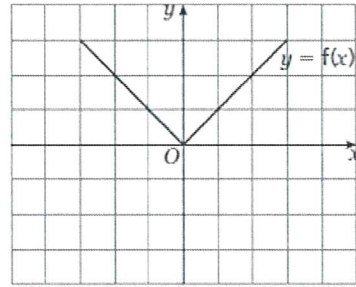
9 The graph shows the function $y = f(x)$. Copy the graph and, on the same axes, sketch and label the graphs of $y = -f(x)$ and $y = f\left(\frac{1}{2}x\right)$.

10 The graph shows the function $y = f(x)$. Copy the graph and, on the same axes, sketch the graph of $y = -f(2x)$.

11 The graph shows the function $y = f(x)$ and a transformation, labelled C. Write down the equation of the translated curve C in function form.

12 The graph shows the function $y = f(x)$ and a transformation labelled C.
Write down the equation of the translated curve C in function form.

13 The graph shows the function $y = f(x)$.

    a Sketch the graph of $y = -f(x)$.

    b Sketch the graph of $y = 2f(x)$.
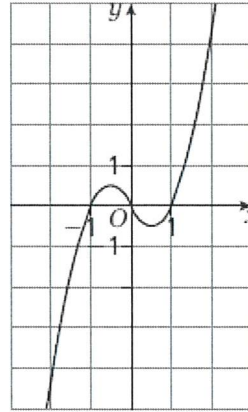
Extend

14   **a** Sketch and label the graph of $y = f(x)$, where $f(x) = (x - 1)(x + 1)$.

    b On the same axes, sketch and label the graphs of $y = f(x) - 2$ and $y = f(x + 2)$.

15   **a** Sketch and label the graph of $y = f(x)$, where $f(x) = -(x + 1)(x - 2)$.

    b On the same axes, sketch and label the graph of $y = f\left(-\frac{1}{2}x\right)$.

Answers

1
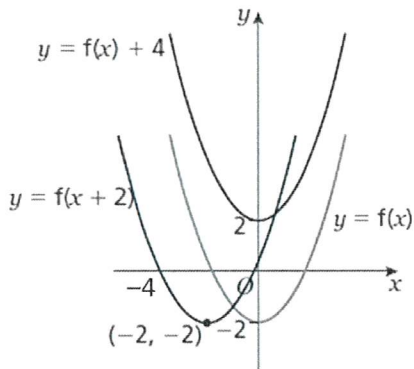
2
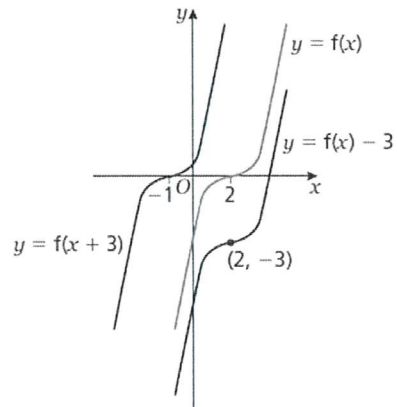
3
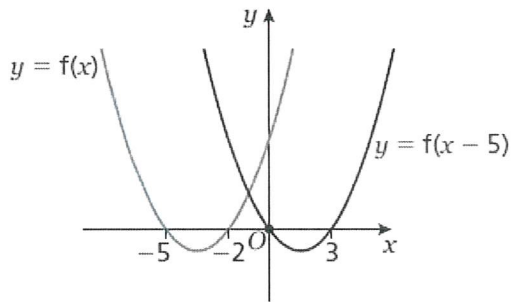
4 C_1: $y = f(x - 90°)$
C_2: $y = f(x) - 2$

5 C_1: $y = f(x - 5)$
C_2: $y = f(x) - 3$
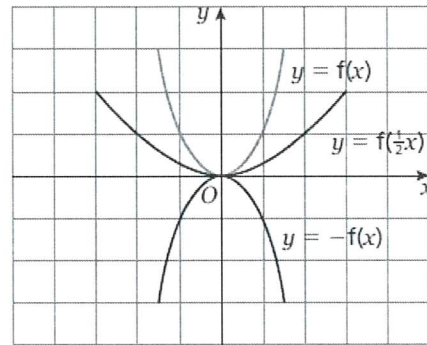
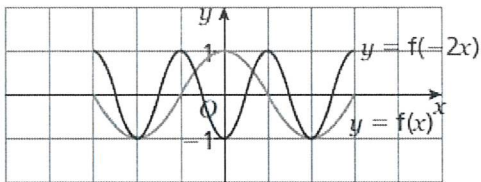**6** **a**
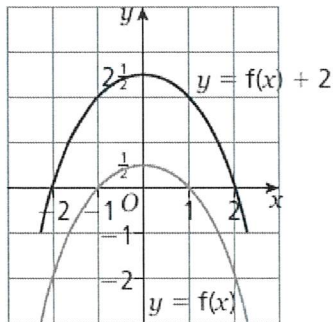
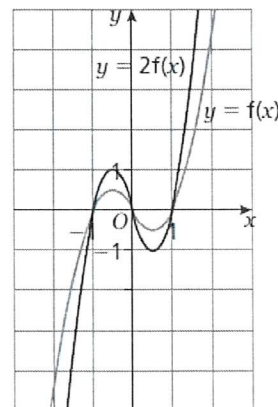

b
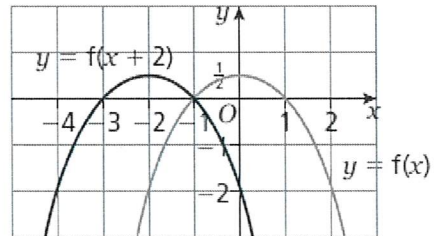
7 a
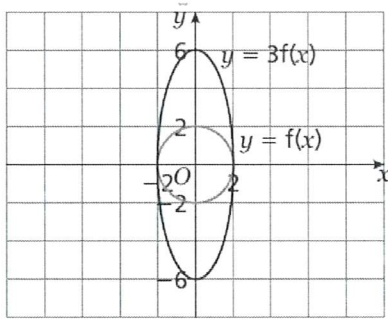
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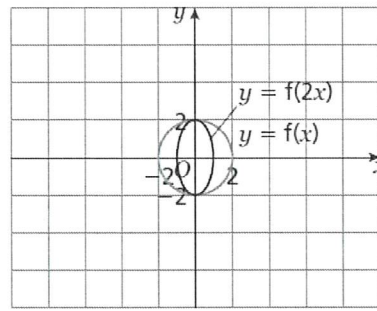
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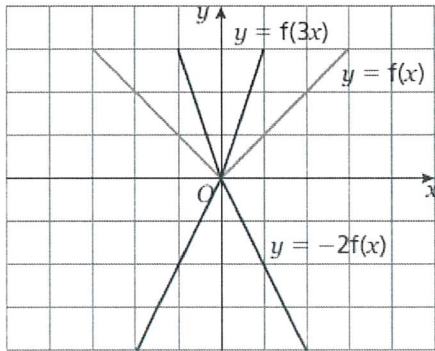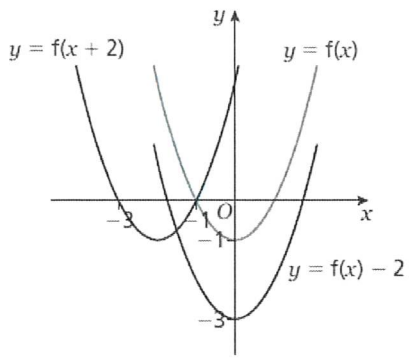

9

10

11 $y = f(2x)$

12 $y = -2f(2x)$ or $y = 2f(-2x)$

13 a

b

14

15